# The Timing of Complementary Innovations

Francisco Poggi*

January 2, 2021

**⇒ Click here for the latest version ⇐**

**Abstract**

This paper studies the development of technologies that require complementary innovations. At each point in time, resources are allocated to research projects that are completed stochastically in the form of breakthroughs. I solve the problem of efficient dynamic allocation of resources by showing that, for complements, this problem is equivalent to an auxiliary recursive problem. In some cases, the solution involves developing the innovations in sequence. In others, it is optimal to develop multiple innovations simultaneously. I provide simple conditions that determine the efficient timing of development. Finally, I analyze when this efficient timing can be implemented with decentralized incentives.

*Northwestern University, Department of Economics. 2211 Campus Drive, Evanston, IL 60208, USA. fpoggi@unorthwestern.edu. I want to thank Jeff Ely, Bruno Strulovici, and Wojciech Olszewski for insightful comments and guidance. I'm also grateful to Eddie Dekel, Yingni Guo, Joshua Mollner, Alessandro Pavan, Ludvig Sinander, Rob Porter, Quitze Valenzuela-Stookey, Asher Wolinsky and Gabriel Ziegler for comments and suggestions, as well to seminar participants at Northwestern University. Preliminary and incomplete. All errors are my own. If you are reading this on an electronic device, click here for the latest version.

# 1 Introduction

The aim of innovation policy is to orient scarce resources toward the most socially valuable R&D projects. The value of an innovation, however, is sometimes tied to the uncertain outcome of other developments: A new treatment for a medical condition is more valuable if there is a way to diagnose the condition at an early stage, when the treatment is more effective; likewise a new diagnosis method is more valuable if there are available treatments for the medical condition; quantum software can only be used if there is quantum hardware to implement it.[1] These complementarities in innovation are important—and increasingly relevant—in many industries.

R&D projects carry high levels of uncertainty, both in terms of outcomes—the project may or may not prove successful—and in terms of costs—it is not clear how much time and how many resources will be required to complete the development. These uncertainty, however, is partially and gradually resolved as the innovations are pursued: some projects are successfully completed, while others are found to be more challenging than what was originally expected. This paper studies the allocation of resources toward the development of complementary innovations. To fix ideas, consider two abstract complementary innovations, $A$ and $B$. The first question is what is the *efficient* way to develop the projects. Should resources be concentrated on the development of $A$ and then be switched to $B$ if and only when $A$ is successfully developed? Or should both $A$ and $B$ be developed in parallel? Moreover, when should a project be abandoned or put on hold? The question of efficient development of innovations is relevant for innovation policy, but also for firms or venture capitalists that want to develop innovations in a way that is consistent with profit maximization.

Moreover, in many industries, R&D is carried out in a decentralized way, by firms or individuals with private interests. Today, we observe startups and established companies alike investing resources to develop quantum software that can only be implemented with hardware that does not yet exist, and it is not clear that it ever will.[2] Is the allocation carried out by private agents efficient? Moreover, how does the environment (for example, the level of concentration of the industry, the appropriability of innovations, or the requirements for patentability) affect the allocation of resources to the development of complementary

[1] Shor's quantum algorithm, a method for solving integer factorization problems in polynomial time, was written in 1994, four years before the first quantum computer prototype was developed.

[2] The current quantum computers seem to be insufficiently powerful to be useful. The highest integer that has been factorized using Shor's algorithm is $21 = 7 \times 3$. Moreover, it is not clear that it will be possible to develop a sufficiently powerful quantum computer, at least in the near future.

innovations?

The first contribution of this paper is to introduce a tractable model that features key aspects of the R&D process when the development of innovations requires non-specific resources such as time, money, or attention. A unit of a resource (attention from now on) is allocated over two complementary projects at each point in time. For each of the projects, a success arrives discretely in the form of a breakthrough. More precisely, a success arrives when the cumulative amount of attention paid to a project reaches a certain level. Successes are observable, but the amount of attention required to succeed is unknown. Moreover, success amounts are independent across projects, so that working on one of the developments does not provide any information about how demanding will the other one be. The only relationship between the projects comes through the complementarity in payoffs.[3] The joint value of the innovations is realized when the development stage is endogenously terminated and depends on the set of projects that was completed by that time. This paper is particularly concerned with complementary innovations, where the marginal value of completing each project is increasing in the set of projects that was successfully completed. Finally, the cost of development takes the form of a constant flow throughout the development stage.

The second contribution of the paper is to introduce techniques to solve the dynamic problem of attention allocation. More precisely, I use properties of the solution to prove an equivalence between the original allocation problem and a simpler recursive problem. For complements, a success in one project makes it more attractive to keep working on the remaining one. Proposition 1 shows that when this is the case, the dynamic problem of attention allocation is equivalent to a recursive problem in which the decision maker pledges a certain amount of attention to each of the remaining projects, and commits to abandon the projects if no success is obtained after the attention is allocated. Thus, all that matters for efficiency is how much total attention the DM is willing to pledge to each project, and not the specific order in which the agent allocates the attention. This is useful because it reduces the strategy space to a few parameters. The intuition is that given the complementarities in payoff, the amount that the planner is willing to work on a project if there is no new success is the minimum she is going to work on that project on any optimal strategy. Since the planner would work this amount independently of the outcome of other projects, *when* she does it is not payoff-relevant.

I analyze specifically the canonical case where projects are perfect complements and the

[3] In the extended model of Section 5.1, the assumption of independence is weakened.

rate of success of each project is constant over time but unknown. The beliefs about these rates evolve with the outcomes of the development process. In particular, lack of success is evidence in favor of the rates being relatively low, or in other words "the project $i$ being relatively more challenging."

If the rate of success for each project were known, the timing of development is irrelevant. Any project that is worth pursuing is worth completing; therefore, the order of completion is not going to affect the final expected payoff.[4] In contrast, when the rate of success is uncertain, the order of development is relevant, since it affects the arrival of information about the unknown parameters. An initial failure to develop $A$ not only reduces the prospects of ever completing $A$, but also decreases the expected returns from completing $B$. The problem therefore does not fit in the classical experimentation framework and thus there is no general result (such as the Gittins index) that governs the optimal dynamic allocation.[5]

Consider the case where project $A$ is of uncertain feasibility, that is, the success rate is either zero or $\lambda_A$, and project $B$ has a known success rate $\lambda_B$. In this case, it is efficient to first work on project $A$: there is no learning by working on $B$, so there is no efficiency loss in back-loading all development of $B$. Front-loading the development of $A$ increases the speed of learning, which is valuable because of the option given by the stopping decision. The intuition from the previous case applies more generally: the efficient allocation of resources reflects the optimal learning process about the potential of the joint project. For symmetric projects, Proposition 2 partitions the parameter space in two. For projects with high uncertainty about the rate of success and high costs, it is efficient to work in a sequence starting with the less promising project. For other parameters, it is optimal to work on the projects simultaneously. This result is generalized to asymmetric projects in Proposition 3.

The final contribution of the paper is the study of inefficiencies that arise due to decentralization and when whether the efficient allocation can be implemented with private incentives. The allocation of private R&D resources depends on several factors: who assigns these resources, the appropriability of the innovations—which is determined by the legal and patent systems—and how informed the agents are about a given project's successes. In Section 4.2, I study private allocation of resources in the extreme case of a decentralized allocation.

A *decentralized allocation* myopically allocates the resources to the projects maximizing the immediate expected return of development. In particular, the decentralized allocation does

[4]A similar logic holds when the completion time for each project is deterministic.

[5]The problem fits the framework of *restless multi-armed bandits*, for which there is no general index solution.

not consider how the allocation changes the dynamics of beliefs about the feasibility and difficulty of the projects.[6] The return from innovation, in turns, captures all the marginal value of the innovation and a portion of the value from subsequent innovations. We consider policies that alter this proportion or *appropriation level*.

There are two inefficiencies associated with decentralization. First, an imperfect appropriability might induce underdevelopment since innovators don't consider the full value of the innovations. This inefficiency is present even fixing the timing of developments. With endogenous timing, a new type of inefficiency might appear, due to inefficient timing of development.

With substitute projects, Bryan and Lemus [2017] show how decentralization and competition biases the allocation of resources toward fast, easy projects to the detriment of harder but cost-efficient ones. Although one might think that this effect is also a concern with complements, Proposition 4 shows that this is not the case: the decentralized allocation of resources is not biased toward projects just because they are thought to be easier. In particular, if the rates of success for both projects have the same support of beliefs, the decentralized allocation of resources is efficient when the appropriation is perfect.

For projects where the beliefs have different supports, the decentralized allocation is inefficient even when appropriability is perfect. In particular, the allocation of resources is biased toward projects where learning is slower. I show that these inefficiencies disappear, however, if the stakes are sufficiently high.[7]

The reminder of the paper is as follows. In the next section, I discuss the relevant literature. Section 2 introduces the model. Section 3 shows when is it possible to solve the allocation problem by looking at a auxiliary problem. In Section 4, we focus on a canonical case of perfect complements with uncertain rates of success. In Section 4.1 the efficient allocation is characterized. In Section 4.2 we identify the inefficiencies generated by the private allocation of resources by a decentralized industry. Section 6 concludes.

[6]The decentralized allocation is equivalent to the equilibrium outcome of an industry that consists of a continuum of firms that race to develop the innovations, and each of whom controls an equal portion of the total unit of resource available at each moment in time. The firms don't consider the informational externalities that their actions generate.

[7]This contrasts again with the case of substitutes, where higher stakes magnify the race effects. See Bryan et al. [2020].

### 1.1 Related Literature

The main contribution is to the literature that studies complementary innovations. Scotchmer and Green [1990] and Ménière [2008] ask what is the optimal inventive requirements for a patent in the context of complementary innovations. Biagi and Denicolò [2014] study the optimal division of profits with complementary innovations. Fershtman and Kamien [1992] study the effects of cross licensing in the incentives to innovate. Bryan and Lemus [2017] study the direction of innovation in a general setting that accounts for complementary innovations. Bryan et al. [2020] focus on the effect of a *crises*—a proportional increase in the payoff from innovations—in the direction of innovation with partial substitutes. In these papers there is no learning in the development stage since the process of information arrival is memoryless.

A particular type of complementary innovation is that sequential developments or cumulative innovation. Papers that study sequential developments include Gilbert and Katz [2011] and Green and Scotchmer [1995]. Moroni [2019] studies a contracting environment with sequential innovations. In these papers the timing of innovation is exogenously given. I focus on complementary innovations in which the timing is determined endogenously by the allocation of resources. To the best of my knowledge, my paper is the first one to combine an endogenous timing of development with a learning process in the development stage in the context of innovation.

Another important related literature studies the problem of dynamic information acquisition from multiple sources. With Poisson information structure, both Nikandrova and Pancs [2018] and Che and Mierendorff [2017] study an agent that acquires information before an irreversible decision. Nikandrova and Pancs [2018] studies the case of independent processes while Che and Mierendorff [2017] studies processes that are negatively correlated. Mayskaya [2019] also studies irreversible decision with Poisson structure in a general setting. Ke and Villas-boas [2019] study problem of independent information sources where the agent learns about the state by observing a Brownian process. Klabjan et al. [2014] study the problem of sequential acquisition of information about the attributes of an object.

Liang et al. [2018] and Liang and Mu [2020] compare the performance of optimal strategies and a different strategy. Liang et al. [2018] asks the question of how well does a strategy that neglects all dynamic considerations and acquire information in a myopic way performs with respect to the optimal information acquisition strategy. Liang and Mu [2020] compare efficient information acquisition to what results from the choices of short-lived agents who do not internalize the externalities of their actions.

The paper shares many key elements with the theory of scheduling in operations research. This literature is concerned with the problem of specifying an order in which jobs or tasks should be completed. Although there are papers in this literature that incorporate uncertainty in the amount of resources that each task demands, the type of objective functions is typically different.

## 2 Model

A decision maker (DM) can work on two research projects, $A$ and $B$. The DM decides when to stop the research activity and, before that, in which way to allocate resources across the projects. Time is continuous and each instant before stopping, the DM allocates a unit of resource across the projects that were not completed so far. $\mathcal{A} := [0,1]^2 \cup a_0$ is set of available actions at each time, where $a_0$ is the stopping action.

Let $\alpha_i(t)$ be the amount of resource allocated to project $i$ at time $t$. Each project is completed when the cumulative resources allocated to it $X(t) := \int_0^t \alpha_i(\tilde{t})\ d\tilde{t}$ reaches a certain amount $\tau_i$. Project completion is observable but $\tau_i$ is unknown. Formally, the agent observes the stochastic process $S(t) := \{i : \tau_i \leqslant X(t)\}$ that represents the set of projects that was completed so far. The completion times of the projects are independent with $F_i$ the cdf of project $i$.

At each point in time, the agent decides either to stop or how to allocate a unit of the resource across the incomplete projects. The resource is scarce: $\sum_{i \in S(t)^c} \alpha_i(t) \leqslant 1$ for all $t$.

If the DM stops at time $t$, the payoff derived from the innovations is $q(S(t))$. As a normalization, if no project was completed the payoff is zero.

**Assumption 1** (free disposal)**.** $q$ is increasing in the inclusion order, i.e.

$$q(S) \leqslant q(S') \quad \text{for all } S \subseteq S'.$$

We are interested in complementary innovations.

**Definition 1.** The projects are *complements* if the function $q$ is supermodular, that is,

$$q(A) + q(B) \leqslant q(\{A, B\})$$

The projects are *perfect complements* if $q(A) = q(B) = 0$.[8]

Finally, using the resources is costly: there is a constant flow cost of $c$ during the development stage, that is independent on which project the DM works on.[9] There is no discounting.[10] Thus, the payoff of an DM that stops at time $t$ and completed projects $S$ by that time is $q(S(t)) - c \cdot t$. The DM is an expected-payoff maximizer.

**Strategies**

A *strategy* is a map from the set of histories $\mathcal{H}$ to the set of actions $\mathcal{A}$. A *stationary strategy* only uses part of the information contained in the history, namely the cumulative resources and the set of completed projects. In particular, in a stationary strategy the actions of the agent do not depend on the order in which resources were allocated so far. Notice that since no resources can be spent on completed projects, by knowing the set of completed tasks $S$ and the vector of cumulative resources $X$, the agent can recover the completion times: $\tau_i = X_i$ for $i \in S$. Let $\mathcal{H}^\circ := 2^{\{A,B\}} \times \mathbb{R}^2_+$ be the set of stationary histories. Formally, a stationary strategy consists on a vector field for each subset of developments.

**Definition 2.** A *stationary strategy* is a function $x : \mathcal{H}^\circ \to \mathcal{A}$ such that $x_i(S, X) = 0$ for all $i \in S$ and

$$\sum_{i \in K \setminus S} x_i(S, X) \leqslant 1 \qquad \forall\ (S, X) \in \mathcal{H}$$

Let $\mathcal{X}$ be the set of stationary strategies.

[8]Substitutes are defined by $q$ being submodular, and perfect substitutes by the property that $q(A) = q(B) = q(\{A, B\})$.

[9]This assumption is innocuous since we can normalize the time unit for different projects, by changing the distribution of $\tau$, so that the cost is the same for all projects.

[10]Alternatively, we could have had a discount factor instead of the linear cost in time. The qualitative features of the solution remain unchanged for this alternative version of the model.

# 3 Order-independence and the Efficient Allocation

The problem of the decision maker is to choose a strategy to maximize their expected payoff. Since all the information about the underlying uncertainty is embedded in the stationary history, it is without loss of optimality to focus on stationary strategies. For an initial state $(S, X)$, strategy $x \in \mathcal{X}$, and vector $\tau$ of completion times, there is a deterministic extra time that the agent that follows strategy $x$ will work $\tilde{T}(x, \tau)$ and deterministic set of projects that will be completed by that time $\tilde{S}(x, \tau)$. The *allocation problem* can be expressed as:

$$V(S, X) \quad = \quad \max_{x \in \mathcal{X}} \quad \mathbb{E}_\tau \left[ q(\tilde{S}(x, \tau)) - c \cdot \tilde{T}(x, \tau) \,\middle|\, (S, X) \right]$$

A different problem is as follows. The agent decides how many resources to *pledge* to each of the remaining projects, and then allocates them in some order. If none of the projects is completed after all the pledged resources are allocated, the agent has to stop. If at least one of the projects is completed, more resources can be allocated to the remaining ones. We are going to refer to this alternative problem as the *order-independent problem.* The name stems from the fact that for a given pledge, the order in which the resources are allocated will not affect the outcome.

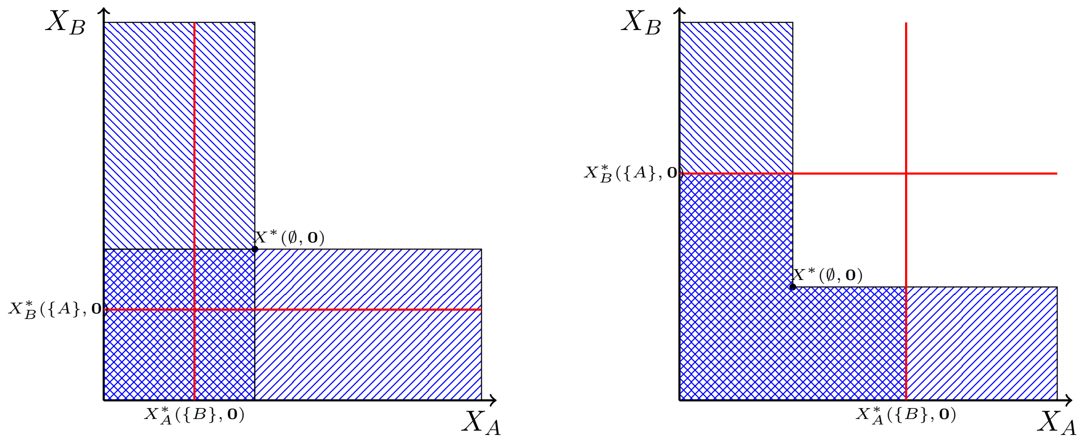
Formally, the order-independent problem can be recursively defined as follows. Let $\hat{S}_X(\tau)$ be the set of completed projects at $X$, that is $\hat{S}_X(\tau) := \{i \in K : \tau_i \leqslant X_i\}$ and $D_{X,\hat{X}}(\tau)$ an indicator function that takes value 1 if there are no new completed projects from $X$ to $\hat{X}$, i.e. $D_{X,\hat{X}}(\tau) := \mathbf{1}_{\left\{\hat{S}_{\hat{X}}(\tau) = \hat{S}_X(\tau)\right\}}$.

$$\hat{V}(S, X) \quad = \quad \max_{\hat{X} \geqslant X} \quad \mathbb{E}_\tau \left[ W(X, \hat{X}, \tau) \ - \ c \ \sum_{i \in K} (\min\{\hat{X}_i, \tau_i\} - x_i) \,\middle|\, (S, X) \right]$$

where $W(X, \hat{X}, \tau) := D_{X,\hat{X}}(\tau) \cdot q(S) + [1 - D_{X,\hat{X}}(\tau)] \cdot \hat{V}(\hat{S}_{\hat{X}}(\tau), \hat{X} \wedge \tau)$.

Let $X^*(S, X)$ be the solution to the order-independent problem. It must be that $V \geqslant \hat{V}$ since the DM can always choose a strategy that implements the solution to the order-independent problem. We are interested in conditions on $q$ and $F$ such that these two problems equivalent, i.e. $V = \hat{V}$.

When completing a project induces the agent to work more on the remaining one, the order in which the agent works on the projects before the first success irrelevant modulo the cumulative work at the stopping decision. This is formalized in the following proposition.




(a) Substitute projects. (b) Complementary projects.

Figure 1: Solution to the order-independent problem. The agent completes project A (project B) when the vector of completion times falls in the area with north east (west) pattern.

**Proposition 1.** If the projects are complements, then $V = \hat{V}$. If the projects are not complements, there exists a family of distributions $\{F_i\}$ such that $V \neq \hat{V}$.

*Proof.* in the Appendix A.1. Here is a sketch that provides the intuition of the result.

For any strategy, there is a (potentially infinite) amount of resources $\hat{X}$ that the agent allocates to each project before stopping, conditional on that no project is ever completed. For complementary projects, the amount the agent allocates on a project for an *optimal* strategy must be less than the amount that the DM would be willing to allocate to the project if the other one was completed: $\hat{X}_i \leqslant X_i^*(\{j\}, X)$. This means that a pledge is not binding: if the agent pledges $\hat{X}$, independently of the outcomes of the process, he was going to put more than $\hat{X}_i$ resources on project $i$.

Figure 1 illustrates the intuition for Proposition 1. When projects are complements, success increases the amount the agent is willing to pledge to a project. So $X_i^*(\emptyset, \mathbf{0}) \leqslant X_i^*(\{j\}, \mathbf{0})$, as in Figure 1b. This implies that the agent never regrets a pledge made. If the projects are substitutes, then $X_i^*(\emptyset, \mathbf{0}) < X_i^*(\{j\}, \mathbf{0})$, as in Figure 1a. At any point $X$ with $X_i > X_i^*(\{j\}, \mathbf{0})$, the agent that completes project $j$ would regret his pledge to project $i$.

Proposition 1 implies that for complements, and only for complements, it is possible to solve the dynamic optimal allocation by finding the *optimal pledge*. To emphasize the generality of the result, notice that no assumption was made on the distributions of completion times $F_i$. Thus, the result holds even for discrete time with arbitrary costs.[11]

Let $h_i := F_i'/(1-F_i)$ be the *completion rate* of project $i$. When the completion rate is decreasing, failures depress the prospects of each project, we can bound the stopping frontier— the set of states at which the agent stops—using the following lemma.

**Lemma 1.** If $h_i$ is decreasing for one of the projects and the projects complements, then $V(\cdot, X)$ is supermodular for all $X^*(\emptyset, X) \neq X$.

*Sketch of the proof:* Since the projects are complements, completion of one of the projects weakly increases the willingness to work on the remaining one. If at $(\emptyset, X)$ the DM wants to stop, it must be that

$$h_j(\emptyset, X)(V(\{j\}, X) - V(\emptyset)) \leqslant c$$

But the agent would be weakly willing to work on project $B$ if project $A$ was completed. Thus, since

$$h_j(\emptyset, X)(V(\{i\}, X) - \underbrace{V(\emptyset, X)}_{=0}) \leqslant c \leqslant h_j(\{i\}, X)(V(\{i,j\}, X) - V(\{i\}, X))$$

By independence, $(V(\{i\}, X) \leqslant (V(\{i,j\}, X) - V(\{i\}, X))$.

Lemma 1 says that supermodularity of $V(\cdot, X)$ holds for all $X$ where the agent wants to stop before the first success. $V(\cdot, X)$ is not supermodular for all $X$. For instance, for $q$ modular we have

$$V(\{A, B\}, X) = q(\{A, B\}) = q(A) + q(B) \leqslant V(A, X) + V(B, X).$$

Where the last inequality holds strictly for all $X$ where the DM would like to continue working on any of the projects. However, since for the modular case the stopping problem can be thought as independent across projects, at a stopping point $v(\{i\}, X) = q(i)$ for both projects.

[11] To see this just consider an $F_i$ with mass probabilities at different times. The difference between the mass points can be interpreted as the cost of working on the project for an extra period.

**Change of variables** To solve the order-independent problem, it is convenient to change variables and work with the probabilities of success. Thus, instead of choosing a pledge, the DM chooses the probability with witch she wants to succeed in each project, conditional on the (lack of) success on the other project. Let $p_i = F_i(\hat{X}_i(\emptyset, \mathbf{0}))$ and $\bar{p}_i = F_i(\hat{X}_i(j, \mathbf{0}))$, the order-independent problem can be written as:

$$\max_{\bar{p}_A, p_A, \bar{p}_B, p_B} (p_A\bar{p}_B + p_B\bar{p}_A - p_A p_B)q(\{A,B\}) + \sum_{i=A,B} p_i(1-\bar{p}_j)q(i) - \sum_{i=A,B} [p_i C_j(\bar{p}_j) + (1-p_i)C_j(p_j)]$$

Where $C_i(p)$ is the expected cost associated with completing a project with probability $p$.

$$C_i(p) := \int_0^{F^{-1}(p)} (1 - F(\tau))c \; d\tau$$

When the hazard rate of project $i$ is decreasing, the cost function is convex, and the solution to the problem can be characterized by the first order conditions: $C_i'(\bar{p}_i) = q(\{A,B\}) - q(j)$ and $C_i'(p_i) = q(i) + \frac{\bar{p}_j - p_j}{1-p_j} q(\{A,B\})$.

Sometimes the solutions to the order-independent problem is such that all resources are pledged to one of the projects. We are going to say that it is efficient to work on the projects *in sequence* if for every $X$ there exists a project $i$ such that $X_i^*(\emptyset, X) - X_i = 0$.

In the next section, we focus on a family of canonical problems and use the results from this section to answer qualitative aspects of the solution—when is it optimal to work on the projects in sequence and when it is not—and to analyze the effects of industry concentration on the allocation of resources.

# 4 Uncertain Rate of Success

The focus of this section is on a set of canonical problems: perfect complements with constant but unknown completion rate. The agent knows that the rate takes one of two possible values, $\lambda_i \in \{\lambda_i^L, \lambda_i^H\}$ and, as before, the rates are independent across projects, with $p_i := \Pr(\lambda_i = \lambda_i^H)$.[12] As resources are allocated to a project and this is not completed, the agent becomes more pessimistic about its difficulty. Let $p_i(X)$ be the probability of the rate for project $i$ being $\lambda^H$ when no project was completed after resources $X$ where allocated. We are going to use $\delta_i := \frac{1}{2}(\lambda_i^H - \lambda_i^L)$ and $\bar{\lambda}_i := \frac{1}{2}(\lambda_i^H + \lambda_i^L)$, and normalize time so that $\frac{1}{2}(\bar{\lambda}_A + \bar{\lambda}_B) = 1$.

[12]Thus, $F_i = 1 - p_i e^{-\lambda_i^H} - (1-p_i)e^{-\lambda_i^L}$.

**Definition 3.** The projects are *symmetric* if $\lambda_A^H = \lambda_B^H$ and $\lambda_L^H = \lambda_B^L$. If projects are symmetric, we say that project $i$ is the *most promissing project* at the state $(\emptyset, X)$ if $p_i(X) > p_j(X)$.

Notice that symmetry is not requiring the initial beliefs $p_A$ and $p_B$ to be the same.

## 4.1 Efficient Allocation

**Observation:** If project $A$ has a known rate of success, it is efficient to work on the projects in sequence. The efficient sequence starts with the project of uncertain rate (project $B$).

The reason is that there is no learning by working on the project with known rate. So, for any strategy with $X_i^*(\emptyset, X) > 0$, the expected return of the same strategy with $X_i^*(\emptyset, X) = 0$ is weakly larger.

The next proposition tell us that the nature of the optimal strategy depends on measure that is increasing in the normalized cost and the uncertainty about the underlying success rate.

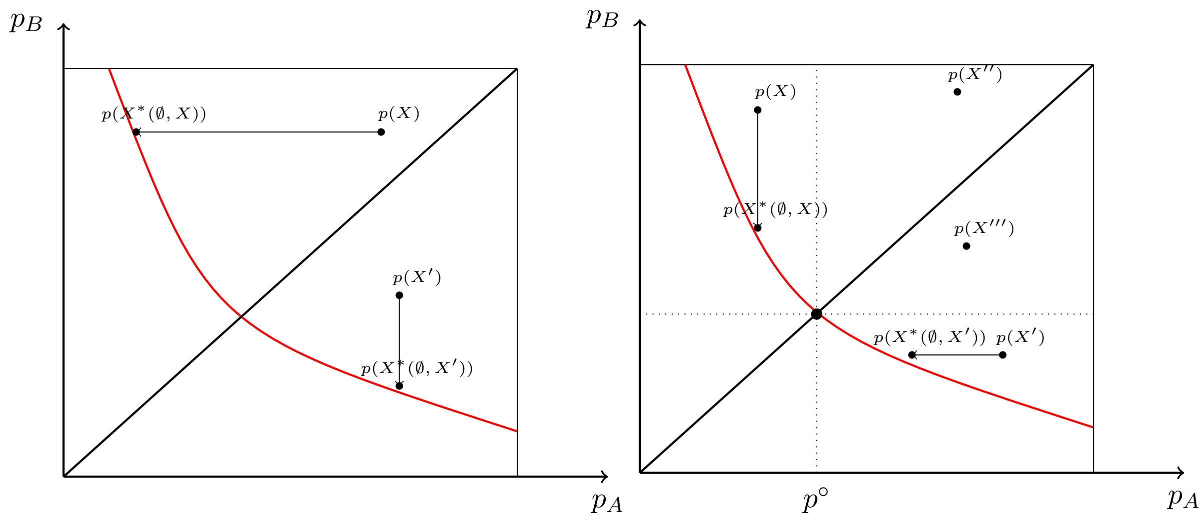
**Proposition 2.** For symmetric projects, let $g := 2\frac{c}{\gamma} + \delta^2$.

- If $g > 1$, it is efficient to work on the projects in sequence, starting from the least promising project.
- If $g < 1$, it is efficient to work more on the more promising project:

$$X_i^*(\emptyset, X) - X_i > X_j^*(\emptyset, X) - X_j \qquad \Leftrightarrow \qquad p_i(X) > p_j(X)$$

  Moreover, there exists a $p^\circ \in (0,1)$ such that if $p_i(X) > p^\circ$ for both projects then $p_i(X^*(\emptyset, X)) = p^\circ$ for both projects. If $p_i(X) < p^\circ$ for one of the projects then it is efficient to work on the projects in sequence, starting with the most promising one.

Figure 2 shows the optimal allocation of resources before the first success for different priors, in belief space. The red curve represents the boundary of the set of points at which the agent is willing to stop before the first success $p(X^*(\emptyset, \mathbb{R}_+^2))$. In Figure 2a, when $g > 1$,




(a) When $g > 1$ it is optimal to work on the projects in sequence, starting from the least promising one.

(b) When $g < 1$ it is optimal to work more on the more promising project.

Figure 2: Optimal allocation for symmetric projects in the first stage.

it is optimal to work only on the project with lower prior. In Figure 2b, when $g < 1$, to the left of the 45° line the initial prior $p_i > p^\circ$ for $i = A, B$, then it is optimal to work on both projects before stopping in the first stage. More precisely, for an initial state $(\emptyset, X)$, $X_i^* = p_i^{-1}(p^\circ) - X$. Thus, there are multiple stationary strategies that are optimal. All these are payoff-equivalent by Proposition 1.

The result says that it is efficient to concentrate the resources (and therefore work in sequence) when the cost of development is sufficiently high, or the difference between the high and low rates is sufficiently large for both projects. The intuition is that in this case, having a single project that is difficult is sufficiently bad to abandon the joint project, so by concentrating the resources the DM gets to learn fast if this is the case. In the cost of development is sufficiently low, or the difference between the high and low rates is low for both projects, then it is optimal to work on the project simultaneously. In this case, having a single project that is difficult is not sufficient to stop.

We can formalize this intuition by interpreting the result in terms of optimal information acquisition. There are three possible states: both projects are easy ($\lambda_A = \lambda_B = \lambda^H$), both are hard ($\lambda_A = \lambda_B = \lambda^L$) or one is easy and the other one hard ($\lambda_A \neq \lambda_B$). For the decision problem to be interesting it must be that the DM would be willing to work on the projects if he knew that both are easy, and he does not want to work on the projects when both of them are hard.

Suppose that the DM would be willing to work on the projects if he knew that one was difficult and the other one was easy. Then the partition of the state space that is relevant for decision making is whether there is at least one easy project (continue) or both projects are hard (abandon).

The probability of the event 'at least one of the projects is easy' is $p^{\text{OR}} = p_A + p_B - p_A \cdot p_B$. By assigning extra resources $dX_i$ to project $i$ and not succeeding, the change in $p^{\text{OR}}$ is

$$\frac{dp^{\text{OR}}}{dX_i} = (1 - p_j)\frac{dp_i}{dX_i} = -p_i(1 - p_i)(1 - p_j)2\delta$$

The fastest way to learn about the relevant state is to work on the project with highest $p$, and therefore to work on the projects simultaneously.

If the DM does not want to work when one of the project is hard and the other one is easy, the relevant state is whether there is a hard project or not. There is no hard project with probability $p^{\text{AND}} = p_A \cdot p_B$. By working on project $i$ for a period $dt$ and not succeeding the change in $p^{\text{AND}}$ is

$$\frac{dp^{\text{AND}}}{dX_i} = p_j \frac{dp_i}{dX_i} = -p_i p_j (1 - p_i)2\delta$$

The fastest way to learn about the relevant state is to work on the project with lowest probability of success, and therefore to work on the projects in sequence.

When does the DM want to continue working when one of the projects is difficult and the other one is easy? When the expected cost of completing both projects is less than the payoff from doing so, i.e.

$$\underbrace{\frac{c}{1+\delta} + \frac{c}{1-\delta}}_{\text{Expected cost}} < \underbrace{\gamma}_{\text{benefit}}$$

Rearranging we can see that this is equivalently to $g < 1$. Proposition 2 can be extended to asymmetric projects as follows.

**Proposition 3.** Let $g_i := 2\frac{c/\gamma}{\lambda_i} + \frac{1}{4}\left(\frac{\delta_i}{\lambda_i}\right)^2$,

1. If $g_i > 1$ for both projects, then it is efficient to work on them in sequence.

2. If $g_i < 1$ for both projects, and $\lambda_i^L > \lambda_j^H$ then it is efficient to work on the projects in sequence, starting with the most promissing one.

3. If $g_i < 1$ for both projects and $\lambda_i^H > \lambda_j^L$ for $i \neq j$, then there exists a $p \in (0,1)$ such that if $p_i(X) > p$ for both projects, then for $X^* = X^*(\emptyset, X)$:

$$\frac{h_A(X^*)}{V(K \setminus \{A\}, X^*)} = \frac{h_B(X^*)}{V(K \setminus \{B\}, X^*)}$$

*Proof.* in the Appendix B.1. Extra conditions for the case where $g_A < 1 < g_B$ can be found in Appendix D.3. □

## 4.2 Decentralized Allocation

So far, we focused on the efficient allocation: how would resources be allocated by a single forward-looking decision maker that internalizes the social value of the innovations. Research and development is rarely carried out by such decision maker. The resources in the economy might be controlled by different agents, with private incentives. The level of concentration and the appropriability of the innovations affect the incentives of the agents and, ultimately, the allocation of resources to different projects.

**Imperfect appropriation** Even fixing a sequence of innovations, decentralization might generate inefficiencies due to imperfect appropriation. The agent who develops an innovation might not capture how his innovation increases the value of subsequent innovations. Efforts to develop an innovation might stop inefficiently early. A way to fix this inefficiency is to give compensate early developers for the subsequent development of complementary innovations.

Compensating innovators for subsequent innovations, however, opens the door to a different type of inefficiency when the timing of development is endogenous. Innovators might be tempted to work on easy innovations to capture rents from the posterior development of difficult complementary innovations. Thus, the decentralized resource allocation might be inefficient when it is efficient to start with the difficult innovations.

**Decentralized allocation** In this section, we consider a family of strategies that, at each state, maximize a flow payoff. The flow payoff depends on the reward that is expected from innovation. The reward function is parametrized by the share of value that appropriation level $\alpha$. These reward functions consider both the current marginal value of the innovation as well as the potential marginal value, in the case that complementary technologies are developed.

**Definition 4.** A strategy $x$ is the *decentralized allocation* for appropriation $\alpha$ if for each state $(S, X)$ maximizes

$$\sum_{i=A,B} x_i[h_i(X) \cdot R_i^{\alpha}(S, X) - c]$$

Where

$$R_i^{\alpha}(S, X) := [\underbrace{(q(S \cup \{i\}) - q(S))}_{\text{immediate}} + \alpha \underbrace{(V(S \cup \{i\}, X) - q(S \cup i))}_{\text{potential}}$$

We can decompose an innovation's worth in two: an *immediate worth*, how useful it is given the current technology, and a *potential worth* in the form of future uses. The parameter $\alpha$ captures the level of appropriability of the subsequent innovation rents. When $\alpha = 0$, the decentralized allocation assigns resources to the project that creates higher expected immediate worth net of the cost of development. When $\alpha > 0$ the allocation internalizes, at least partly, the potential uses of the innovation in the future.[13]

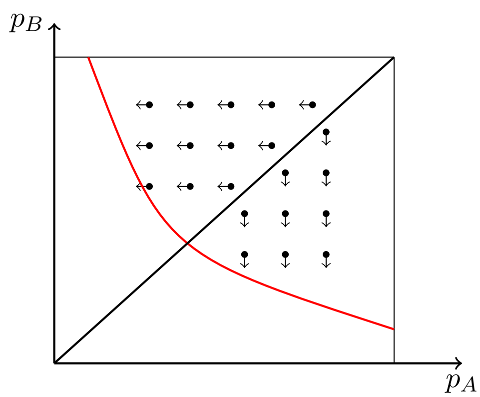
The following result when is the allocation efficient for symmetric projects.

**Proposition 4.** If the projects are symmetric, the decentralized allocation is efficient if and only if $\alpha = 1$.

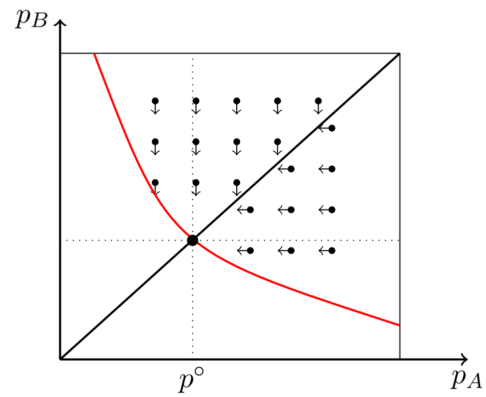
When $\alpha = 1$, the decentralized allocation chooses to work always on the project with highest hazard-to-value ratio $\frac{h_i(X)}{V(\{j\},X)}$, which is constant on $X_j$. We know, from Proposition 2, that when projects are symmetric and $g > 1$, it is efficient to work on them in sequence, always starting with the least promising one. $g > 1$ implies that the hazard rate $h_i(X)$ decreases faster than the value $V(\{j\}, X)$ so for $X'_j = X_j$ and $X'_i > X_i$

$$h_i(X)V(\{i\}, X) \geqslant h_j(X)V(\{j\}, X) \qquad \Rightarrow \qquad h_i(X')V(\{i\}, X') \geqslant h_j(X')V(\{j\}, X')$$

[13]The decentralized allocation with $\alpha = 1$ turns out to be the equilibrium allocation in a game with a continuum of firms that compete in the development stage when the first firm to succeed gets the whole surplus of the development process. This micro-foundation is explored in Appendix C.




(a) When $g > 1$ the decentralized allocation works always on the least promising project.




(b) When $g < 1$ the decentralized allocation works always on the most promising project.

Figure 3: Decentralized allocation for $\alpha = 1$ with symmetric projects in the first stage.

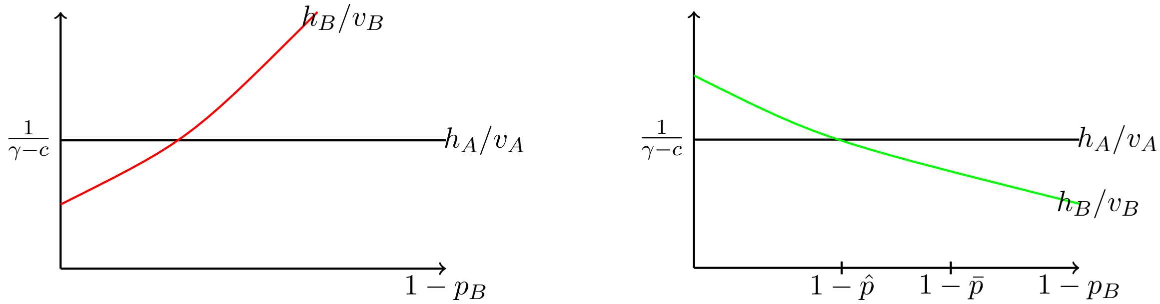
and this implies that the decentralized allocation for $\alpha = 1$ also works on the projects in sequence starting from the least promising one.

Proposition 4 indicates that, for symmetric projects, the intuition that was developed at the beginning of this section is flawed: there is no trade-off between the inefficiencies caused by imperfect appropriability and the ones caused by the endogenous timing of development. The reason is that when the easy project is completed, the remaining project is difficult for which there is a smaller chance of success. Thus the value of completing an easy project is smaller than the value of completing a difficult one, what compensates for the differential in the rate of success.

For asymmetric projects however, the value of completing a more difficult project might not perfectly compensate for the lower probability of success. The next proposition shows that, even for asymmetric projects, whenever it is efficient to work more on the more promising project, the decentralized allocation of resources is efficient.x

**Proposition 5.** If $g_i < 1$ for both projects, the decentralized allocation is efficient for $\alpha = 1$.

When $g_i < 1$ the value decreases faster than the hazard rate. Proposition 5 shows that




(a) **Case I**: $\lambda_B^H < \lambda_A$.

(b) **Case II:** $\lambda_B^H > \lambda_A$ and $\bar{p} < \hat{p}$.

Figure 4: The decentralized allocation is inefficient if $h_B/v_B$ is lower than $h_A/v_A$ for a belief for which it is efficient to work.

when this holds for both projects, it is efficient to work always on the project with highest hazard-to-value ratio. This is exactly what the decentralized strategy does.

The decentralized allocation does not consider the information generated by the allocation of resources. Decentralization will thus bias the allocation of resources towards projects where there is less learning. The next proposition considers the case where project $A$ has a known rate of success. For project $B$ the rate that is unknown.

**Proposition 6.** If project $A$ has a known success rate, the decentralized allocation for $\alpha = 1$ is inefficient if and only if $\lambda_B^H < \lambda_A$ and $p_B$ is large enough.

*Proof.* The decentralized allocation allocates resources to the project with highest hazard-to-value ratio, where $v_i(X_i) = V(K \setminus \{i\}, X)$. The observation from below says that it is efficient to work in sequence starting from project $B$.

Lets call $\hat{p}$ the posterior at which is efficient to stop working (interior if the proble is not trivial). $h_A/v_A$ is constant and equal to $\lambda_A/(\gamma - c/\lambda_A)$.

If $h_B/v_B$ is below $h_A/v_A$ when the belief is close to 1, then there is an initial belief $p$ such that it is optimal to work on $B$ but the decentralized allocation works instead on $A$. This is ilustrated in Figure 4a. $h_B/v_B$ when $p$ is close to 1 is $\lambda^H/(1 - c/\lambda^H)$, so the condition is

$$\frac{\lambda^H}{1 - c/\lambda^H} \leqslant \frac{\lambda_A}{\gamma - c/\lambda_A} \qquad \Leftrightarrow \qquad \lambda^H \leqslant \lambda_A$$

If $h_B/v_B$ is higher than $h_A/v_A$ at $p = 1$, the only way there could be an inefficiency is if $h_B/v_B$ is decreasing and the stopping belief $\bar{p}$ is higher than the belief at which $h_B/v_B$ and $h_A/v_A$ intersect, $\hat{p}$. This situation is ilustrated in Figure 4b. This, however, is never the case:

By Lemma 1, at the efficient stopping point $X^*$, $V(\{A\}, X^*) + V(\{B\}, X^*) \leqslant \gamma$. But $V(\{B\}, X^*) = \gamma - \frac{c}{\lambda_A}$, so $V(\{A\}, X^*) < c/\lambda_A$. Moreover at the stopping point, $h_B \cdot V(\{B\}, X^*) = c$, so

$$h_B(X^*) \cdot V(\{B\}, X^*) = c \geqslant \underbrace{h_A(X^*)}_{\lambda_A} \cdot V(\{A\}, X^*)$$

Rearranging we get

$$\frac{h_B(X)}{V(K \setminus \{B\}, X)} > \frac{h_B(X^*)}{V(K \setminus \{B\}, X^*)} \geqslant \frac{h_A(X^*)}{V(K \setminus \{A\}, X^*)} \qquad \forall X : X_B < X_B^*$$

Where the first inequality holds since $g_B < 1$. □

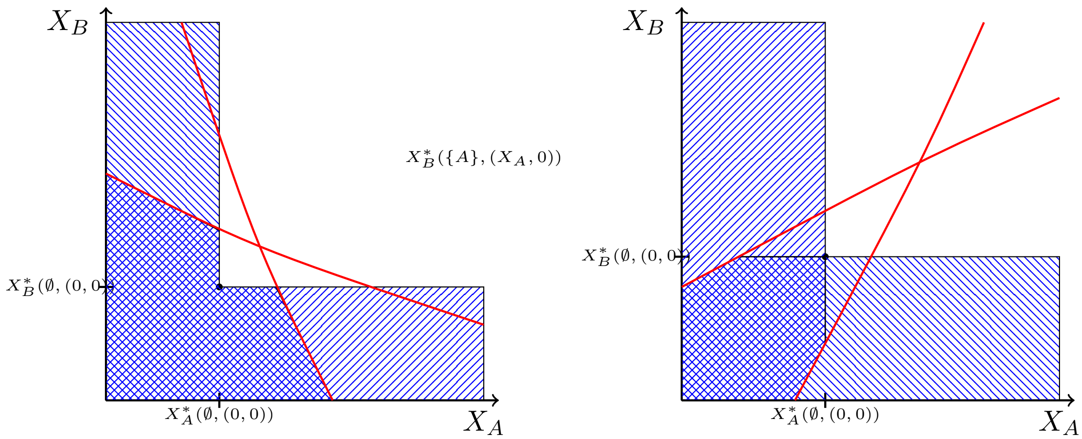
The intuition that if a project is thought to be easier this would attract more attention to it is partially flawed. As the previous result shows, inefficiencies will show up if one of the projects has a higher rate of success than the other in every state, but they also require that the efficient to start project is thought to be as relatively easy.

# 5 Extensions

In this section, I explore potential departures from the model introduced in Section 2. In particular, I extend results relaxing independence, considering more than two projects.

## 5.1 Relaxing Independence

So far we assumed that the projects are independent. Proposition 1 can be generalized more generally to affiliated projects. Let the vector of completion times $\tau$ be drawn from a distribution with density function $f$.




(a) Positively affiliated projects.

(b) Negatively affiliated projects.

Figure 5: Dependent completion times. The agent completes project A (project B) when the vector of completion times falls in the area with north east (west) pattern.

**Definition 5.** The projects are *affiliated* if for every $\tau, \hat{\tau} \in \mathbb{R}^2_+$

$$f(\tau \wedge \hat{\tau}) \cdot f(\tau \vee \hat{\tau}) > f(\tau) \cdot f(\hat{\tau})$$

**Proposition 7.** If the projects are complements and affiliated, then $V = \hat{V}$.

The result of the proposition hinges on the success in one project incentivizing the agent to work more on the remaining one. For projects that are affiliated, a success in one of them is good news about the distribution of the completion times of the other, what makes it more attractive to continue working on it.

If the completion times are not affiliated, like in the Figure 5b, it can be that a very early success in one of the projects leads to optimally lowering the amount that the DM wants to work on the remaining project. Thus, pledging could be costly and the two problems are not equivalent.

## 5.2 More than two projects

The DM can work on a finite set of projects $K := \{1, 2, ..., k\}$. As before, the DM allocates a unit of resource across the projects that were not completed so far $S(t)^c$. Again, we define complementarity by the supermodularity of the value of innovations.

**Definition 6.** Projects in the set $K$ are *complements* if $q$ is supermodular

$$q(A \cup B) + q(A \cap B) \geqslant q(A) + q(B) \qquad \forall A, B \subseteq K$$

If the hazard rate is weakly increasing for all projects, the allocation problem and the order-independent problem.

**Proposition 8.** If the projects are independent and $h_i$ is weakly increasing for all $i \in K$ then $V = \hat{V}$ for all $q$.

The reason is that when the hazard rate is increasing, if a project is worth allocating any resource, then it must be worth completing. Thus, for the optimal strategy, the set of completed projects is the same for all realizations of $\tau$.

For more than two projects, however, we cannot claim that the problems are equivalent even with complementary, affiliated projects. The reason is that complementarity of the projects does not guarantee monotonicity of the solution to the problems that was used to prove the equivalence with two projects.

**Claim 1.** For $k > 2$, independent and complementary projects is *not* sufficient for $\hat{V} = V$.

We show this claim by means of the following counterexample:

**Example 1.** Let $K = \{A, B, C\}$. Suppose $q(\{A, B\}) = \gamma < q(\{A, B, C\}) = 1$. $q(S) = 0$ for any subset. And suppose $C$ is either feasible or infeasible, and that you can learn instantly about it. $\lambda_L^C = 0, \lambda_H^C = \infty$. The optimal strategy is to learn about $C$, and then doing the optimal thing for $A$ and $B$ (that might be different depending on whether $C$ is completed or not).

In the case where

$$c < \frac{\lambda_L \lambda_H}{\lambda_L + \lambda_H} < \frac{c}{\gamma}$$

then by results when $C$ is completed it is optimal to work simultaneously, $Y_i(\{C\}, 0) > 0$ for $i = A, B$. But when $C$ fails, it is optimal to work in sequence, so $Y_i(\emptyset, 0) = 0$ for $i \in \{A, B\}$.

# 6 Conclusion

Innovation is one of the main determinants of long-term economic growth. Thus, understanding the trade-offs in different approaches to innovation as well as the inefficiencies associated with economic environments is of central importance.

This paper makes substantive contributions to the understanting of the problem of development of complementary innovations:

First, the problem of efficient development of complementary innovations features different challenges than that of substitute innovations: failures in one development affect the expected returns from complementary innovations. With complementary innovations, however, successes make it more attractive to continue working on the remaining developments, what simplifies the problem substantially.

Second, allocating resources to innovation projects in an efficient way involves developing complementary innovations with a specific timing: sometimes it is efficient to develop in sequence and sometimes it is efficient to develop multiple innovations simultaneously. Sequential development is more likely to be efficient for high cost of development and higher uncertainty about the completion rates of the projects.

Third, an important part of innovation is carried out by the private sector. The timing of innovation is partly determined by the investment decisions of agents whose objectives are typically misaligned from the social welfare. Complementary innovations, generate investment dynamics that are different than for substitutes. In particular, the allocation of resources is not simply biased towards the easy and fast component to the detriment of the hard but cost-effective ones.

# Bibliography


Biagi, A. and V. Denicolò (2014). Timing of discovery and the division of profit with complementary innovations. *Journal of Economics and Management Strategy 23*(1), 89–102.

Bryan, K. A. and J. Lemus (2017). The direction of innovation. *Journal of Economic Theory 172*, 247–272.

Bryan, K. A., J. Lemus, and G. Marshall (2020). Crises and the Direction of Innovation.

Che, Y.-k. and K. Mierendorff (2017). Optimal Sequential Decision with Limited Attention.

Fershtman, C. and M. I. Kamien (1992). Cross licensing of complementary technologies. *International Journal of Industrial Organization 10*(3), 329–348.

Gilbert, R. J. and M. L. Katz (2011). Efficient division of profits from complementary innovations. *International Journal of Industrial Organization 29*(4), 443–454.

Green, J. R. and S. Scotchmer (1995). On the Division of Profit in Sequential Innovation. *RAND Journal of Economics 26*(1), 20–33.

Ke, T. T. and J. M. Villas-boas (2019). Optimal learning before choice. *Journal of Economic Theory 180*, 383–437.

Klabjan, D., W. Olszewski, and A. Wolinsky (2014). Attributes. *Games and Economic Behavior 88*, 190–206.

Liang, A. and X. Mu (2020). Complementary Information and Learning Traps. *The Quarterly Journal of Economics 135*(1), 389–448.

Liang, A., X. Mu, and V. Syrgkanis (2018). Optimal and myopic information acquisition. *ACM EC 2018 - Proceedings of the 2018 ACM Conference on Economics and Computation*, 45–46.

Mayskaya, T. (2019). Dynamic Choice of Information Sources.

Ménière, Y. (2008). Patent law and complementary innovations. *European Economic Review 52*(7), 1125–1139.

Moroni, S. (2019). Experimentation in Organizations.

Nikandrova, A. and R. Pancs (2018). Dynamic project selection. *Theoretical Economics 13*(1), 115–143.

Scotchmer, S. and J. Green (1990). Novelty and Disclosure in Patent Law. *RAND Journal of Economics 21*(1), 131–146.

# A Omitted proofs from Section 3

## A.1 Proof Proposition 1

The proof is based on Lemma 2 that is stated for $k$ projects. First, some preliminaries.

For any stationary strategy $x \in \mathcal{X}$ and initial state $(S, X_0)$, there is a *trajectory* $y_S : \mathbb{R}_+ \times \mathbb{R}^k \to \bar{\mathbb{R}}^k$ that is the (unique) solution to the differential equations

$$\begin{cases} \nabla y_S(t, X_0) = x(S, y_S(t, X_0)) \\ \quad y_S(0, X_0) = X_0 \end{cases}$$

We will refer to $Y(S, X_0) = \lim_{t\to\infty} y_S(t, X_0)$ as the abandonment point of the strategy $x$ given an initial state $(S, X_0)$.

**Definition 7.** A strategy has *increasing abandonment points* if

$$Y(S, X_0) \leqslant Y(\hat{S}, X_0) \text{ for all } S \subseteq \hat{S}.$$

**Definition 8.** Two strategies $x, \tilde{x}$ *have the same abandonment points* if for each initial state, the abandonment point is the same for both strategies.

**Lemma 2.** If two strategies $x, \tilde{x}$ have the same abandonment points, and these abandonment points are increasing then the two strategies have the same expected payoff.

*Proof.* The proof works by induction. The Lemma trivially holds for $k = 1$. Assume that it holds for $k = 1, 2, ..., m - 1$, we want to show that it holds for $k = m$.

Consider strategies $x, \tilde{x}$ and a initial state $(\emptyset, X_0)$. Let $Y(\emptyset, X_0)$ be the associated abandonment point. For each set $S \neq \emptyset$, the continuation problem is analogous to one with less than $n$ projects, so the lemma holds. Let $V(S, X)$ be the value of the two strategies at the state $(S, x)$ for $S \neq \emptyset$.

Consider a strategy $z$ with the same abandonment points than $x$ and such that for any $S \neq \emptyset$, $z(S, X) = x(\emptyset, X)$ for all $X$ with $x(\emptyset, X) \neq 0$. We can do this since $Y(S, X_0) \geqslant Y(\emptyset, X_0)$. For any $\tau$, the new strategy has the same payoff than the original: either no project is successful and both abandon at the same point or the same project is succesful at the same point, and the continuation value is the same.

Similarly, we can construct a strategy $\tilde{z}$ with the same abandonment points but such that for any $S \neq \emptyset$, $\tilde{z}(S, X) = \tilde{x}(\emptyset, X)$ for all $X$ with $\tilde{x}\emptyset, X) \neq 0$. $\tilde{z}$ and $\tilde{x}$ shield the same payoff. We end the proof by showing that $z$ and $\tilde{z}$ must also have the same payoff for any realization of the success times $\tau \in \mathbb{R}^k_+$.

Let $\bar{S} = \{i \in K : \tau_i < Y(\emptyset, X_0)\}$, that is the set of projects which completion time is below the abandonment point. Both $z$ and $\tilde{z}$ reach $Y(\emptyset, X_0)$ with probability one. The payoff is therefore

$$V(\bar{S}, Y(\emptyset, X_0)) - c \sum_{i \in \bar{S}} \tau_i - c \sum_{i \notin \bar{S}} Y_i(\emptyset, X_0)$$

Taking expectation over the realization of $\tau$ completes the proof. $\square$

The intuition for Lemma 2 is the following: if the abandonment is increasing, then the current abandonment point is the least attention you are willing to put on the remaining projects by the end of the day. Since the attention it is going to be paid eventually, the order in which the agent does it is not gonna determine the outcome.

**Proposition 1.** Consider $k = 2$. If the projects are complements, then $V = \hat{V}$. If the projects are not complements, there exists a family of distribution $\{F_i\}$ such that $V \neq \hat{V}$.

**($\Leftarrow$)** We want to show that $q$ supermodular implies that any strategy that has the same abandonment points than an optimal strategy is also optimal.

**Lemma 3.** For two complementary projects, any optimal strategy has increasing abandonment points.

*Proof.* By contradiction, assume that $Y_i(j, X_0) < Y_i(\emptyset, X_0)$. Then, $\square$

**Lemma 4.** For two affiliated and complementary projects, any optimal strategy has increasing abandonment points.

*Proof.* We want to prove that for any optimal strategy $x$, $Y_i(j, X_0) \geqslant Y_i(\emptyset, X_0)$. By supermodularity of $q$, the marginal value of $i$ is larger when $j$ was completed than when it is not. If it is optimal to work on project $i$ when it is not clear if $j$ is going to be completed or not, it must be optimal to work on $i$ when $j$ was already completed.

Formally, by contradiction assume $Y_i(j, X_0) < Y_i(\emptyset, X_0)$. Then there is a time $t$ such that $y_{\emptyset,i}(t, X_0) = Y_i(j, X_0)$. Let $X := y_\emptyset(t, X_0)$. Consider the trajectory starting at $(\emptyset, X)$ associated with strategy $x$. If this strategy was copied starting on the state $(j, Y_i(j, X_0))$ with a dummy project $j'$ that starts at $x_j$ the expected continuation payoff must be negative (otherwise it is worth continuing at $Y_i(j, X_0)$).

$$q(j) + \int_{Y_i(j,X_0)}^{Y_i(\emptyset,X_0)} \frac{1 - F_i(\tau_i|X_0)}{1 - F_i(Y_i(j, X_0))} \left[h_i(\tau_i)(q(ij) - q(j)) - c\right] \, d\tau_i \geqslant$$

$$q(j) + \int_0^T \frac{1 - F(Y(j, X_0) + y_\emptyset(\tau, X) - X)}{1 - F(Y(j, X_0))} \alpha_i(\tau_i) h_i(\tau_i)(q(ij) - q(j)) \, d\tau$$

$$- \int_{Y_i(j,X_0)}^{Y_i(\emptyset,X_0)} \frac{1 - F_i(\tau_i|X_0)}{1 - F_i(Y_i(j, X_0))} c \, d\tau \geqslant$$

$$\int_0^T \frac{1 - F(Y(j, X_0) + y_\emptyset(\tau, X) - X)}{1 - F(Y(j, X_0))} \left[\alpha_i(\tau_i) h_i(\tau_i)(q(ij) - q(j)) + \alpha_j(\tau) h_j(\tau) q(j)\right] \, d\tau$$

$$- \int_{Y_i(j,X_0)}^{Y_i(\emptyset,X_0)} \frac{1 - F_i(\tau_i|X_0)}{1 - F_i(Y_i(j, X_0))} c \, d\tau \geqslant$$

$$\int_0^T \frac{1 - F(Y(j, X_0) + y_\emptyset(\tau, X) - X)}{1 - F(Y(j, X_0))} \left[\alpha_i(\tau_i) h_i(\tau_i) q(i) + \alpha_j(\tau) h_j(\tau) q(j)\right] \, d\tau$$

$$- \int_{Y_i(j,X_0)}^{Y_i(\emptyset,X_0)} \frac{1 - F_i(\tau_i|X_0)}{1 - F_i(Y_i(j, X_0))} c \, d\tau \geqslant$$

$$V(\emptyset, Y(\emptyset, X)) \geqslant 0$$

But this strategy shields more than the continuation at $X$ thus the project should stop $X$, so $X = Y(\emptyset, x_0)$ leading to a contraction. $\square$

Using Lemma 2, any strategy that has the same abandonment points than $x$ is gonna get the same expected payoff and therefore be optimal.

**($\Rightarrow$)** : The proof works by contrapositive. If $q$ is *not* supermodular, there are cdfs $\{F_i, F_j\}$ such that $Y_i(j, X_0) < Y_i(\emptyset, X_0)$.

*Proof.* Since $q$ is not supermodular, $q(\{i\}) > q(\{i,j\}) - q(\{j\})$. Let $F_i = 1 - e^{-\lambda_i x}$ with $\lambda_i$ such that

$$q(\{i\}) > \frac{c}{\lambda_i} > q(\{i,j\}) - q(\{j\})$$

and let $j$ never succeed, i.e. $F_j = 0$. Rearranging we have that

$$\lambda q(\{i\}) - c > 0 > \lambda(q(\{i,j\}) - q(\{j\})) - c$$

What implies that for any $X_0$, $Y_i(\emptyset, X_0) = \infty$ and $Y_i(\{j\}, X_0) = X_0$. $\square$

## A.2 Proof of Lemma 1

*Proof.* $h_i$ decreasing implies that $V(S, \cdot)$ is decreasing for all $S$.

If $X = X^*(\{A\}, X) = X^*(\{B\}, X)$, then $V(S, X) = q(S)$ and by complementarity of K, $V(\cdot, X)$ is supermodular. If $X \neq X^*(\{i\}, X)$ for some $i$,

$$\frac{\partial V(K \setminus \{i\}, X)}{\partial X_i} = c - h_i(X_i)\left[V(K, X) - V(K \setminus \{i\}, X)\right] \leqslant 0$$

At any point $X \in X^*(\emptyset, \mathbb{R}^k_+)$ it must be that $c \geqslant h_i(x_i) \cdot V(\{i\}, X)$. So,

$$h_i(X_i)V(\{i\}, X) - h_i(X_i)\left[V(\{A, B\}, X) - V(K \setminus \{i\}, X)\right] \leqslant 0 \qquad \forall X \in X^*(\emptyset, \mathbb{R}^k_+)$$

Rearranging,

$$V(\{B\}, X) + V(\{A\}, X) \leqslant V(\{A, B\}, X) \qquad \forall X \in X^*(\emptyset, \mathbb{R}^k_+)$$

$\square$

# B Ommitted proofs from Section 4

## B.1 Proof of Propositions 2 and 3

### Some preliminaries

Let $\delta_i$ be $\lambda_i^H - \lambda_i^L$. Using Bayes' rule, the beliefs $p_i(X)$ evolve

$$p_i(X) = \frac{p_i e^{-\delta_i x_i}}{(1 - p_i) + p_i e^{-\delta_i x}}$$

As the agent becomes more pessimistic, the subjective hazard rate $h_i(t)$ becomes lower.

$$h_i(X) = \lambda_L^i + p_i(X)\delta_i$$

Notice that

$$g_i > 1 \quad \Leftrightarrow \quad \frac{\lambda_i^L \cdot \lambda_i^H}{\lambda_i^L + \lambda_i^H} > c$$

We prove the alternative:

**Proposition 3':**

1. If $g_i > 1$ for both projects, then it is optimal to work on them in sequence.

2 If $g_i < 1$ for both projects, then the greedy strategy is optimal.

The proof of the proposition is split in three lemmatas. Lemma 5 proves that $g_i$ controls the monotonicity of project $i$'s hazard-to-value ratio. Lemma 6 shows that when both projects have an increasing hazard-to-value ratio, it is efficient to work on them in sequence. Lemma 7 shows that when both projects have a decreasing hazard-to-value ratio the greedy strategy is efficient.

**Lemma 5.** $h_i/v_i$ is monotone. Moreover, $\text{sgn}((h_i/v_i)') = \text{sgn}(g_i - 1)$.

*Proof.* First we show that the monotonicity of $h/v$ depends on weather the value $v$ is higher or lower that an expression $R$.

$$\text{sgn}((h_i/v_i)') = \text{sgn}(h_i' v_i - h_i v_i')$$

$$= \operatorname{sgn}(h_i' v_i - h_i(c - h_i(1 - v_i)))$$

$$= \operatorname{sgn}\left(\underbrace{\frac{h_i(h_i - c)}{h_i^2 + h_i'}}_{R(t)} - v_i\right)$$

Change of variables. In belief space, the concavexity of $R$ is determined by weather $g_i$ is larger or lower than one.

$$\hat{R}'(p) = \frac{2\delta^2 \lambda_L \lambda_H (\lambda_L \lambda_H - c(\lambda_L + \lambda_H))}{(\lambda_L^2 + p\delta(\lambda_L + \lambda_H))^3}$$

$$= M(g - 1)$$

Now we consider two cases: $\lambda_L < c$ and $\lambda_L \geqslant c$.

**Case I:** $\lambda_L \geqslant c$ In this case, the agent would never stop. The value is linear in the beliefs.

$$v(p) = 1 - p\frac{c}{\lambda_H} - (1 - p)\frac{c}{\lambda_L}$$

Since $v(0) = R(0)$ and $v(1) = R(1)$,

$$g > 1 \qquad \Leftrightarrow \qquad v(p) < R(p) \qquad \forall p \in (0, 1)$$

**Case II:** $\lambda_L < c$ In this case, the agent abandons if sufficient time passes with no success. $v$ is convex (information is valuable) and $R$ is concave:

$$\lambda_L < c \qquad \Rightarrow \qquad \frac{\lambda_H}{\lambda_L + \lambda_H}\lambda_L < c \qquad \Leftrightarrow \qquad g_i > 1$$

Since $v(1) = R(1)$ and $v(\hat{p}) = R(\hat{p})$ where $\hat{p}$ is the stopping belief.

$$v(p) < R(p) \qquad \text{for any } p \in (\hat{p}, 1)$$

□

**Lemma 6.** If $h_i/v_i$ is strictly increasing for $i = A, B$, it is optimal to work on the projects in sequence.

*Proof.* By contradiction. Assume that $x := Y(\emptyset, x_0) > 0$. Let $r_i(t) := \frac{h_i(t)}{v_i(t)}$ and $g_i(t) := \frac{h_i'(t)}{v_i'(t)\cdot r_i(t)}$. Since $x$ is an interior stopping point, it must be that $h_A(x_A)v_B(x_B) = h_B(x_B)v_A(x_A) = c$.

Abusing notation I write $f_i$ instead of $f_i(x_i)$. First we show that $r_A' + r_B' > 0$ implies $\frac{h_A' v_A}{h_A v_A'} \cdot \frac{h_B' v_B}{h_B v_B'} < 1$.

$$
\begin{aligned}
r_A' + r_B' > 0 \qquad &\Leftrightarrow \qquad \frac{h_A' v_A - h_A v_A'}{v_A^2} + \frac{h_B' v_B - h_B v_B'}{v_B^2} > 0 \\
&\Leftrightarrow \qquad \frac{h_A v_A'}{v_A^2}\left(\frac{h_A' v_A}{h_A v_A'} - 1\right) + \frac{h_B v_B'}{v_B^2}\left(\frac{h_B' v_B}{h_B v_B'} - 1\right) > 0
\end{aligned}
$$

For all $(x_A, x_B)$ such that $h_A v_B = h_B v_A = c$,

$$
\frac{h_A v_A'}{v_A^2} = \frac{h_B v_A'}{v_B v_A} = \frac{h_A v_B'}{v_A v_B} = \frac{h_B v_B'}{v_B^2}
$$

Where the first and last equality use $h_A/v_A = h_B/v_B$ and the intermediate one uses that $h_B v_A' = h_B(c - h_A(1 - v_A)) = -h_B h_A(1 - v_A - v_B)$ (since $c = h_A v_B$) and equal to $h_A v_B'$ by symmetry. So,

$$
\begin{aligned}
r_A' + r_B' > 0 \qquad &\Leftrightarrow \qquad \frac{h_A v_A'}{v_A^2}\left[\left(\frac{h_A' v_A}{h_A v_A'} - 1\right) + \left(\frac{h_B' v_B}{h_B v_B'} - 1\right)\right] > 0 \\
&\Leftrightarrow \qquad \left[\frac{h_A' v_A}{h_A v_A'} + \frac{h_B' v_B}{h_B v_B'}\right] < 2 \\
&\Leftrightarrow \qquad \frac{h_A' v_A}{h_A v_A'} \cdot \frac{h_B' v_B}{h_B v_B'} < 1
\end{aligned}
$$

Where the second implication uses that $v_A$ is decreasing. And the last one is that the sum of two positive numbers being less than two implies that the product is less than one.

The determinant of the Hessian for the value function $V(\emptyset, x)$ is

$$
\det(H) = (1 - F_A)(1 - F_B)[h_A' h_B' v_A v_B - h_A h_B v_A' v_B']
$$

So

$$\det(H) < 0 \qquad \text{iff} \qquad \frac{h'_A v_A}{h_A v'_A} \cdot \frac{h'_B v_B}{h_B v'_B} < 1$$

And $\det(H) < 0$ rules out the candidate as an optimum (saddle point). □

**Lemma 7.** If $h_i/v_i$ is strictly decreasing for $i = A, B$, the greedy strategy is optimal.

*Proof.*

$$r_i \searrow \quad \Leftrightarrow \quad h'_i v_i - h_i v'_i < 0$$

$$\Leftrightarrow \quad \frac{h'_i v_i}{h_i v'_i} > 1$$

So,

$$r_A \searrow \text{ and } r_B \searrow \quad \Rightarrow \quad \frac{h'_A v_A}{h_A v'_A} \cdot \frac{h'_B v_B}{h_B v'_B} > 1$$

This implies that there is at most one interior candidate for solution ($h_A v_B = h_B v_A = c$), and that if it exist it is the actual solution. We consider the two cases.

**Case I: there is an interior solution candidate.** Then this is the actual solution. Since at the solution $r_A = r_B$ and the $r_i$ are decreasing, by working always on the project with highest $r_i$, the point is eventually reached.

**Case II: there in no interior solution candidate.** Then it must be that $h_i v_j = c \Rightarrow h_j v_i > c$. Thus, the solution is to work in sequence starting with project $j$. Moreover, $h_j/v_j > h_i/v_i$ for all $x$ such that $h_i v_j \geqslant c$, so the greedy strategy also works in sequence starting with $j$. □

# C Equilibrium

There is a continuum of agents, $m \in [0, 1]$. Each agen decides, at each instant, what project to work on $\alpha_m(t) \in \{A, B\}$. Once all agents stop developing, the value of the joint development is split across the agents. The payoff of an agent will depend on what innovations are successfully developed, who developed the innovations, and the timing of development.

We take a reduced-form approach to the problem, with focus on the first stage: let $W_i(X)$ be the expected payoff that is captured by the first agent to innovate when the innovation is $i$ and the state was $(\emptyset, X)$.

**Definition 9.** A *first-stage strategy* for agent $m \in [0,1]$ is a function $s_m : \mathbb{R}_+^2 \to \{A, B, \emptyset\}$.

We assume that the problem is non-trivial: if both projects are easy then it is efficient to work on them, and it is efficient to abandon immediately if both projects were known to be difficult.[14]

A concern is that with complements, the competition in development stage will bias the allocation of resources toward the projects that can be developed faster, and that these will be developed first even if it is efficient to leave these projects for later.

The payoff for an agent $m$ that was working on a project $i$ at the moment in which this project was completed is proportional to the value of that project $V(\{i\}, X)$ and inversely proportional to the mass of individuals working on that project at that point. Let $\pi(s, Y, x)$ be the expected payoff of following first-stage strategy $s$ when the evolution of the process is $Y$ and the aggregate strategy is $x$.

**Definition 10.** A *first-stage stationary equilibrium* consists of a first-stage strategy for each agent, an industry allocation $x : \mathbb{R}_+^2 \to [0,1]^2$ and an evolution of cumulative resources $Y : \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_+^2$ such that

1. Each agent maximizes his expected payoff given the evolution of resources.

$$s(X) \in \arg\max_{s' \in \mathcal{S}} \quad \pi(s, Y(\cdot, X), x(X))$$

[14] In terms of the parameters the condition is

$$\frac{\lambda_A^L \cdot \lambda_B^L}{\lambda_A^L + \lambda_B^L} < \frac{c}{\gamma} < \frac{\lambda_A^H \cdot \lambda_B^H}{\lambda_A^H + \lambda_B^H}.$$

2. The industry allocation aggregates all individual strategies.

$$x(X) = \int_0^1 s_m(X)\ dm$$

3. The evolution of resources is consistent with the allocation of the industry.

$$\begin{cases} \nabla Y(t, X) = x(Y(t, X)) \\ \quad Y(0, X) = X \end{cases}$$

**Lemma 8.** Any stationary equilibrium satisfies

$$x_i(\emptyset, X) > 0 \qquad \Rightarrow \qquad h_i(X)V(\{i\}, X) \geqslant \max\{c, h_j(X)V(\{j\}, X)\}$$

*Proof.* in the Appendix C.1. □

Patent races might introduce distortions.

**Definition 11.** There is a *race effect* when there is no equilibrium whose allocation is efficient.

## C.1 Proof of Lemma 8

*Proof.* Individual profits are

$$\int_0^t \underbrace{\frac{1 - F_A(Y_X(\tau))}{1 - F_A(X_A)} \cdot \frac{1 - F_B(Y_X(\tau))}{1 - F_B(X_B)}}_{\text{Pr(reach } \tau)} \left[ \underbrace{x_{s(Y_X(\tau))}(X) \cdot h(Y_X(\tau))}_{\text{rate success at } \tau} \cdot \underbrace{\frac{V(s(Y_X(\tau)), Y_X(\tau)}{x_{s(Y_X(\tau))}(X)}}_{\text{expected payoff if successful}} - c \right] d\tau$$

Since the individuals do not have a marginal effect on the trajectory and take it as given, the way to optimize the individual profits is every instant.

$$s \in \arg \max_{s' \in \{A,B,\emptyset\}} \quad \left\{ h_{s'}(Y_X(\tau))V(s', Y_X(\tau)) - c \right\}$$

□

# D Extensions: proofs

## D.1 Proof of Proposition 8

*Proof.* Suppose that

$$V(S,x) = \hat{V}(S,x) = \max_{\hat{S}\in 2^{K\setminus S}} \left\{ q(S\cup\hat{S}) - c\sum_{i\in\hat{S}} \lambda_i^{-1} \right\}$$

□

## D.2 Discrete time

In this appendix we will consider the discrete time case $T = \{1, 2, 3, ..., \infty\}$. At any time before stopping the agent decides which project to work on $\alpha_t \in \{A, B, \emptyset\}$. Let $F_i$ be the distribution of successes for project $i$, and $h_i : T \to [0,1]$ the respective hazard rate. Finally, let $v_i : T \to [0,1]$ the value of the joint project when only project $i$ is incomplete as a function of the time spent working on project $i$.

$$v_i(x_i) := q(j) + \max_{T\geqslant x_i} \left\{ \sum_{x=x_i+1}^{T} \frac{1-F(x)}{1-F(x_i)} \left[h(x)(1-q(j)) - c\right] \right\}$$

**Proposition 9.** $h_i/v_i$ decreasing for both projects implies that the greedy strategy is efficient.

*Proof.* Grab an optimal abandonment point $x^* := Y(\emptyset, 0)$ and a trajectory to it. The trajectory has to be greedy at the time before the abandonment point. Otherwise, the optimality of $x^*$ is violated.

Consider now a greedy trajectory and the point $(x_L, x_B^*)$ in that trajectory where crosses $x_B = x_B^*$ (the rightmost one). If the optimum is to the right of the path ($x_A^* > x_L$) then by optimality,

$$\frac{h_A}{v_A}(x_L) \geqslant \frac{h_i}{v_i}(x_i^* - 1) \geqslant \frac{h_j}{v_j}(x_j^*)$$

If $x_L = x_i^* - 1$ then the first inequality holds with equality and there is a greedy path to the optimum: the one we considered changing at the indifferent point $(x_L, x_A^*)$. If

$x_L < x_i^* - 1$ then by strict monotonicity of $h/v$ the inequality holds strictly, what would violate greediness of the strategy at $(x_L, x_A^*)$ $\square$

**Proposition 10.** *$h/v$ increasing for both tasks implies that the efficient allocation is in sequence.*

*Proof.* Suppose that the optimal stopping point $x^* = Y(\emptyset, 0)$ is interior, i.e. $x^* > 0$. Since last period is myopically optimal for each trajectory,

$$\frac{h_A}{v_A}(x_A^* - 1) \geqslant \frac{h_B}{v_B}(x_B^*) > \frac{h_B}{v_B}(x_B^* - 1)$$

$$\frac{h_B}{v_B}(x_B^* - 1) \geqslant \frac{h_A}{v_A}(x_A^*) > \frac{h_A}{v_A}(x_A^* - 1)$$

Where the strict inequalities come from the $h/v$ being increasing for both projects. Thus, a contradiction. $\square$

### D.3 One h/v increasing and one decreasing

**Lemma 9.** If the horizontal sum of the two $h/v$ is increasing, then it is optimal to develop the projects in sequence.

*Proof.* Consider $q(y) := (h_A/v_A)^{-1}(y) + (h_B/v_B)^{-1}(y)$ decreasing for all $y \in R := (h_A/v_A)([0, \bar{t}]) \cap (h_B/v_B)([0, \bar{t}])$. Taking the derivative this implies that

$$\frac{1}{(h_A/v_A)'((h_A/v_A)^{-1}(y))} + \frac{1}{(h_B/v_B)'((h_B/v_B)^{-1}(y))} < 0 \qquad \forall y \in \mathbb{R}$$

$$\frac{(h_A/v_A)'((h_A/v_A)^{-1}(y)) + (h_B/v_B)'((h_B/v_B)^{-1}(y))}{(h_A/v_A)'((h_A/v_A)^{-1}(y)) \cdot (h_B/v_B)'((h_B/v_B)^{-1}(y))} < 0 \qquad \forall y \in \mathbb{R}$$

$$(h_A/v_A)'((h_A/v_A)^{-1}(y)) + (h_B/v_B)'((h_B/v_B)^{-1}(y)) > 0 \qquad \forall y \in \mathbb{R}$$

Or, in other words: $r'_A(x_A) + r'_B(x_B) > 0$ for all points $(x_A, x_B)$ with $h_A(x_A)/v_A(x_B) = h_B(x_B)/v_B(x_B)$. We can use the same logic used in the proof of Lemma 6 to rule out interior points. □

### D.4 Imperfect complements

$\lambda_L > c/(1-q)$ then the agent would never stop. The value is independent of $q$ and linear. The monotonicity of $h/v$ is equivalent to the case where $q = 0$.

Consider now $\lambda_L \in (c, c/(1-q))$. There is a belief at which the agent stops.

$$\hat{p} = \frac{c/(1-q) - \lambda_L}{\delta}$$

If $R(\hat{p}) > v(\hat{p}) = q$ and $R$ is concave, $h/v$ is increasing.

$R(\hat{p}) > q$

$$\frac{c^2 q}{(1-q)[c(\lambda_L + \lambda_H) - (1-q)\lambda_L \lambda_H]} > q$$

Interesting case: $[c(\lambda_L + \lambda_H) - (1-q)\lambda_L\lambda_H] > 0$.

$$\left(\frac{c}{(1-q)}\right)^2 \geqslant \frac{c}{(1-q)}(\lambda_L + \lambda_H) - \lambda_L \lambda_H$$

$$\frac{c}{(1-q)}\left(\frac{c}{(1-q)} - \lambda_L\right) \geqslant \lambda_H \left(\frac{c}{(1-q)} - \lambda_L\right)$$

$$\frac{c}{(1-q)} \geqslant \lambda_H$$

But if this is the case, then the agent does not wish to work on the development even when sure that it is relatively easy.