The Timing of Complementary Innovations

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Abstract

This paper investigates the optimal allocation of attention towards the dynamic completion of interrelated projects, where the final payoff is determined by the set of completed projects and the total attention allocated over time. An allocation policy consists of a stopping decision—determining how much attention is invest before abandoning the projects—and a timing decision—determining the order of investment across projects. We establish a partial order in the set of allocation policies and show that the expected payoff is increases in such order. Moreover, we provide sufficient conditions for the optimal policy to be *timing-independent*. We apply these results to characterize the optimal attention allocation policy in the canonical problem of two perfectly complementary projects with an unknown constant rate of completion.

1 Introduction

The development of innovative products often involves completing multiple intermediate steps. In some cases, these steps must be completed in a specific order, and the

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problem of an innovator boils down to choosing the intensity of work at each step. This problem has been extensively studied in the literature on sequential innovation (e.g. in Gilbert and Katz [2011] and Green and Scotchmer [1995]). However, when these steps can be completed in any order, the problem becomes more complex, as there is an additional strategic consideration of the innovator: she must not only decide how intensively to work, but also which steps to prioritize. This dual challenge remains underexplored in the existing literature, despite its relevance in a wide range of applications.

In this paper, we introduce a framework that combines optimal timing and stopping decisions. A decision maker (DM) engages in various projects that are completed through observable breakthroughs, though the exact amount of resources required to complete each project is unknown. The DM allocates a fixed amount of resources—*attention*—per unit of time across projects and must decide when to stop. The final payoff depends on both the set of completed projects and the total amount of resources allocated.

Consider two projects that are perfect substitutes: upon completion of one project, the additional value gained from completing the remaining one is zero, making it optimal to stop further investments immediately. In this scenario, any resources allocated to the uncompleted project become *effectively wasted*, meaning that, in hindsight of the information obtained, the DM recognizes that a better choice would have been to allocate all resources solely to the project that was eventually completed. The only allocation policies that avoid effective wastefulness entirely are those that allocate resources exclusively to one of the projects. These policies, however, are not necessarily optimal.

For general interdependence of the projects' payoffs, we introduce a partial order of allocation policies based on the concept of effective wastefulness and we show that policies that induce less effective waste perform better in expectation. This result provides a structure that allow us to simplify the problem of finding optimal policies. Furthermore, when considering independent projects that are complements—meaning that the final payoff is supermodular in the set of projects completed—we show that the optimal allocation policy is completely free of effective waste. This result is *tight* in the sense that when projects are not complements, it is possible to construct information structures for which the optimal allocation involves effective waste.

We apply these results to the problem of perfect complementary projects with uncertain completion rates and find that the DM prioritizes projects based on their perceived difficulty. When encountering a difficult project is sufficient to render the entire enterprise unprofitable, the DM prioritizes the project that is perceived to be more difficult. Conversely, if having at least one easy project is enough to make the whole enterprise profitable, the DM prioritizes the project that is perceived to be easier.

Related Literature Weitzman's and Gittins' classical work—Weitzman [1979] and Gittins [1979]—provide a foundational study on timing and stopping rules. Their main result consist on the construction of indices that indicate the optimal order in which alternatives should be explored and when the process should be stopped to maximize expected payoffs. In their framework, the value derived from the exploration process depends solely on the maximum value among the explored alternatives (or "boxes"). In contrast, in this paper, we consider a framework that accounts for dependencies across payoffs, allowing us to adress situation where the value of a successful innovation depends on the interaction with other successes.

This paper contributes to the literatures on learning and scheduling. In the sequential learning literature, a recent series of paper explores the question of how attention should be allocated across different information sources before making an irreversible decision. Gossner et al. [2021] study a decision maker that learns about two substitute alternatives and stops according to a simple rule, focusing on the effect of directing attention toward one item in the final decision. Nikandrova and Pancs [2018], Che and Mierendorff [2019], and Mayskaya [2019] study problems of sequential information acquisition with a Poisson information structure. While Nikandrova and Pancs [2018] focuses on acquiring information about indenpendent alternatives, Che and Mierendorff [2019] analyze a framework in which the different sources reveal information about the same alternative. Mayskaya [2019], on the other hand, considers a decision maker who acquires information about two potentially independent alternatives. Although all these papers focus on information acquisition for an eventual decision, this paper considers a decision maker who allocates attention to projects until they are either completed or they decide to stop.

Departing from the Poisson structure, Ke and Villas-boas [2019] study a problem involving independent information sources, where the DM learns about the state by observing a Brownian process. Klabjan et al. [2014] study the problem of sequential acquisition of information regarding multiple attributes of a signle alternative. Two papers Liang et al. [2018] and Liang and Mu [2020] compare the performance of optimal strategies with alternative approaches. Liang et al. [2018] asks the question of how well a strategy that neglects all dynamic considerations and acquires information in a myopic way performs compared to the optimal information acquisition strategy. Liang and Mu [2020] compare efficient information acquisition to what results from the choices of short-lived agents who do not internalize the externalities of their actions.

The problem of incentives in the context of complementary innovations has been extensively studied. For example, Scotchmer and Green [1990] and Ménière [2008] study optimal inventive requirements for patents in the context of complementary innovations. Bryan and Lemus [2017] study the direction of innovation in a general setting that accounts for complementary innovations. In contrast, this paper, focuses on processes of innovation that involve learning about the difficulty of the projects.

Some complementary innovations are sequential or cumulative. Papers that study sequential developments include Gilbert and Katz [2011] and Green and Scotchmer [1995]. Moroni [2019] studies a contracting environment with sequential innovations. In these papers, the timing of innovation is exogenously given. In contrast, this paper focuses on complementary innovations in which the timing is determined endogenously by the allocation of resources to the projects. To the best of my knowledge, this paper is the first one to combine an endogenous timing of completion with learning.

The paper shares many key elements with the theory of scheduling in operations research. This literature is mostly concerned with the problem of specifying the order in which jobs or tasks should be completed. Although there are papers in this literature that incorporate uncertainty in the amount of resources that each task demands, the objective functions are typically different. A classical question in this literature is how to complete a certain set of tasks in the least possible expected time. In this paper, instead, the set of tasks that end up being completed is endogenous.

The remainder of the paper is structured as follows: Section 2 introduces the model and provides some preliminary analysis. Section 3 discusses the optimal allocation of resources. In Section 4 and Section 5, we present the main results. Finally, in Section 6, we apply these results to characterize the optimal attention allocation when projects have a constant but unknown rate of completion.

2 Setup

There are two projects, A and B. A decision maker (DM) has a unit of attention per unit of time and must decide how to allocate this attention between the two projects. Time is continuous. Let $\alpha_i(t)$ denote the amount of attention allocated to project i at time t. Attention is scarce: $\alpha_A(t) + \alpha_B(t) \leqslant 1$ for all $t \geqslant 0$. Project i is completed when the cumulative attention allocated to it $x_i(t) := \int_0^t \alpha_i(\tilde{t})\ d\tilde{t}$ reaches a certain *completion amount* τ_i. These completion amounts τ_A and τ_B are random variables with support in $\mathcal{T}_A$ and $\mathcal{T}_B$ respectively, where $\mathcal{T}_i$ is a closed subset of the positive reals.

Upon stopping, the DM receives a reward $q(Y)$, which depends on the set Y of completed projects. In addition, the DM incurs costs based on the total attention allocated to each projects. Specifically, if the DM stops at time T after completing projects Y, the final payoff is given by $q(Y) - c_A \cdot x_A(T) - c_B \cdot x_B(T)$, where c_i is the unit cost of attention for project i. The DM aims to maximize its expected final payoff.[1] The reward function q is assumed to be non-decreasing with respect to the set inclusion order and is normalized such that the reward of completing no projects $q(\emptyset)$ is zero.

For expository purposes, we assume that the completion times τ_A and τ_B are jointly continuous, as defined next.[2] Let $G(x_A, x_B) := \Pr(\tau_A > x_A \ \&\ \tau_B > x_B)$ be the joint survival function of completion times.

Definition 1. *Projects are* jointly continuous *if there exists a density function* $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ *such that*

$$G(x_A, x_B) = \int_{x_B}^{\infty} \int_{x_A}^{\infty} f(\tilde{x}_A, \tilde{x}_B)\ d\tilde{x}_A\ d\tilde{x}_B \tag{1}$$

Allocation Policies

A *strategy* is a map that indicates the attention allocation for each possible history. Since all payoff-relevant information in any history consists of (i) the completion status of each project, (ii) the attention allocated to each uncompleted project, and

[1] In this model there is no discounting. Thus, the payoff of the DM depends on the total amount of attention allocated to each project and not the order in which this attention was allocated. The qualitative features of the solution remain unchanged in a model where the DM discounts the future as long as the cost of attention allocation is the same for both projects, i.e. $c_A = c_B$.

[2] However, most of the main results do not rely on this assumption.

(iii) the completion amounts of completed projects, we can, without loss of generality, focus on *allocation policies* that are measurable in the *payoff-relevant state*, defined as a tuple (Y, x, T), where $Y \subseteq \{A, B\}$ indicates the set of completed projects, $x \in \mathbb{R}_+^2$ is the attention allocated to each project, and $T \subseteq \mathbb{R}_+^2$ is the set of completion states that was not discarded.[3]

When a project is completed, its completion amount is perfectly revealed. For an uncompleted project, the DM only knows that the completion amount is larger than the cumulative attention already allocated. Moreover, we assume without loss of generality that it is not possible to allocate attention to completed projects. Thus, in any payoff-relevant state we have that $T = T_A \times T_B$ where T_i is a singleton $T_i = \{x_i\}$ iff $i \in Y$ and $T_i = (x_i, \infty)$ if $i \notin Y$. Note that T is perfectly pinned down by (Y, x), for which we drop the dependence.

Definition 2. *An* allocation policy $\sigma = (\sigma_A, \sigma_B)$ *is a pair of right-continuous functions* $\sigma_i : \mathcal{H} \to [0, 1]$ *satisfying* $\sigma_A(h) + \sigma_B(h) \leqslant 1$ *for all* $h \in \mathcal{H}$.

This paper is concerned with finding the allocation policy that maximizes the expected payoff of the DM. Because there is no reason to backload attention, it is without loss of optimality to focus on policies that either allocate all attention or no attention, effectively stopping. Let $\mathcal{S}$ denote the set of all such allocation policies. For any allocation policy $\sigma \in \mathcal{S}$, we define $V(\sigma)$ as the expected payoff, at time zero, of the DM that follows policy σ.

Definition 3. *For any allocation policy* $\sigma \in \mathcal{S}$ *and set of completed projects* $Y \subseteq \{A, B\}$*, we define* stopping regions *as*

$$S_Y^\sigma := \{x \in \mathbb{R}_+^2 : \sigma(Y, x) = 0\}.$$

In words, the stopping region indicate the states in which the policy stops allocating attention to the projects. For example, after completing project B, the DM continues allocating attention to project A until the cumulative attention x_A is such that $(x_A, \tau_B) \in S_{\{B\}}^\sigma$.

Before any project is completed, a policy σ allocates attention according to $\sigma(\emptyset, x)$. For each allocation policy, there is a corresponding *first-stage policy* $\alpha = (\alpha_A, \alpha_B)$

[3]This simplification holds because the order in which the attention was allocated so far is payoff-irrelevant.

which specifies how attention is allocated over time, conditional on that no project is completed.

3 Optimal allocation policy

In this section, we study the optimal allocation of attention across the two projects. We begin by studying a simplified version of the problem involving a single project, which serves as a benchmark. This simpler setting focuses solely on the stopping decision, making it easier to solve. The solution to this benchmark proves to be useful in characterizing the solution to the more complex two-project problem, which we approach backwards: we first address the *second stage*, where one of the projects has already been completed, and determine the optimal stopping decision for the remaining project. Then, we turn to the *first stage*, where neither project is completed, to characterize the optimal allocation of attention between the two projects.

3.1 Benchmark: single-project problem

As a benchmark, we first consider a simplified setting involving a single project. This setting is defined by three key elements: the reward Π obtained upon completing the project, the cost c incurred for allocating attention to the project, and a distribution of completion times characterized by the cumulative distribution function F.

The DM must decide how much attention to allocate to the project at each point in time. If the DM stops after allocating x units of attention, the final payoff is given by $\mathbb{1}_{\{x \geqslant \tau\}} \cdot \Pi - c \cdot x$, where $\mathbb{1}_{\{x \geqslant \tau\}}$ is an indicator function that equals 1 if the project is completed and 0 otherwise.

As we argued in the previous section, for any history in which the project was not completed, the only payoff-relevant information is the total amount of attention allocated to the project so far. Thus, a *single-project policy* can be defined without loss of optimality as a map $\alpha : \mathbb{R}_+ \to \{0, 1\}$ that indicates the attention allocated to the project as a function of the total cumulative allocation already invested, conditional on the project not being completed yet. A single-project policy is *optimal* if it maximizes the expected continuation payoff of a DM who follows it, for every initial state. Formally, the *expected continuation payoff* π of a DM that has already allocated x

units of attention to the project without completion is defined as

$$\pi(x', x) := [F(x') - F(x)] \cdot \Pi - c \cdot W(x', x)$$

Where $W(x', x) := \mathbb{E}[\min\{\tau, x'\} - x \mid \tau > x]$, is the expected additional attention the DM allocates when they have already allocated x and are willing to allocate up to x'. The problem then reduces to choosing the stopping point x' that maximizes the expected continuation payoff. We define the *continuation value* as $v(x) := \sup_{x' \geqslant x} \quad \pi(x', x)$.

In making a stopping decision, the DM trades off the probability of eventually completing the project $(F(x') - F(x))$ against the expected remaining cost of attention, which is proportional to the expected remaining attention to be allocated, $W(x', x)$.

Given that the action of the DM facing a single-project problem is binary—either to stop or to continue allocating attention—it can be characterized by the states at which the DM would choose to stop.

Definition 4. *We define the* stopping set *of the single-project problem as* $S^* := \{ x \in \mathbb{R}_+ \quad : \quad v(x) = 0 \}$. *The* strict stopping set *of the project is defined as* $S^{**} := \{x \in \mathbb{R} : \forall x' > x, \quad \pi(x', x) < 0\}$.

Naturally, stopping sets and optimal single-project policies are related. We formalize the relationship in the following remark.

Remark 1. *For a single-project problem* (Π, c, F)*, the single-project policy* α *is optimal iff* $\alpha(x) = 0$ *for all* $x \in S^{**}$ *and* $\alpha(x) = 1$ *for all* $x \notin S^*$.

The next result establishes basic comparative statics for single-project problems. Specifically, Proposition 1 asserts that if the reward Π is higher—relative to the cost c of attention—and the hazard rate of project completion is higher, then the DM's optimal stopping set shrinks, implying that the DM would continue to allocate attention for longer to the project for every initial state.

Proposition 1. *Consider two single-project problems* (Π, c, F) *and* $(\hat{\Pi}, \hat{c}, \hat{F})$ *with respective stopping sets* S^* *and* $\hat{S}^*$*. If* $\hat{\Pi}/\hat{c} \geqslant \Pi/c$ *and* F *hazard-rate dominates* $\hat{F}$*, then* $\hat{S}^* \subseteq S^*$.

Proof. In Appendix A.1. □

This intuitive result is based on proving that if it is profitable for the DM to continue allocating attention to a project (Π, c, F), then it will also be profitable to do so for a project $(\hat{\Pi}, \hat{c}, \hat{F})$. Furthermore, when projects have decreasing hazard rate, the optimal allocation policy involves continuing to allocate attention as long as the hazard rate remains higher than the c/Π, which ensures that the expected flow reward exceeds the cost of attention.

Example 1. *Suppose that the hazard rate $h = F'/(1-F)$ is continuous and strictly decreasing. Then, $S^* = \{x \in \mathbb{R}_+ : h(x) \cdot \Pi \geqslant c\}$. Denoting $\bar{h}$ the limit of $h(x)$ as $x \to \infty$, which exists since h is decreasing and bounded below by zero, we get that S^* is as follows:*

$$S^* = \begin{cases} \emptyset & \textit{if} \quad c/\Pi \leqslant \bar{h} \\ \mathbb{R}_+ & \textit{if} \quad c/\Pi > h(0) \\ [h^{-1}(c/\Pi), \infty) & \textit{if} \quad c/\Pi \in (\bar{h}, h(0)) \end{cases}$$

3.2 Two-project problem

We now turn to the studying the two-project problem, which is characterized by the joint distribution of completion times, represented by the survival funciton G, the reward function q, and the attention cost parameters c_A and c_B. As is standard, we approach the problem backwards, starting from states where one of the projects has already been completed.

Second-stage policies

In this section, we study the continuation problem face by the DM after one of the projects is completed. Once project i is completed, the DM's decision boils down to determining when to stop allocating attention to the remaining project. A second-stage policy, therfore, must specify the stopping decision of the DM for each possible realization of the completion time of the completed project.

Remark 2. *For a two-project problem, the optimal attention allocation after a history where project A was completed at τ_A and x_B attention was allocated to project B coincides with the optimal allocation of attention of single-project problem (Π, c, F) where $\Pi = q(\{A, B\}) - q(\{A\})$, $c = c_B$, and $F(x) = 1 - G_1(\tau_A, x)/G_1(\tau_A, 0)$.*

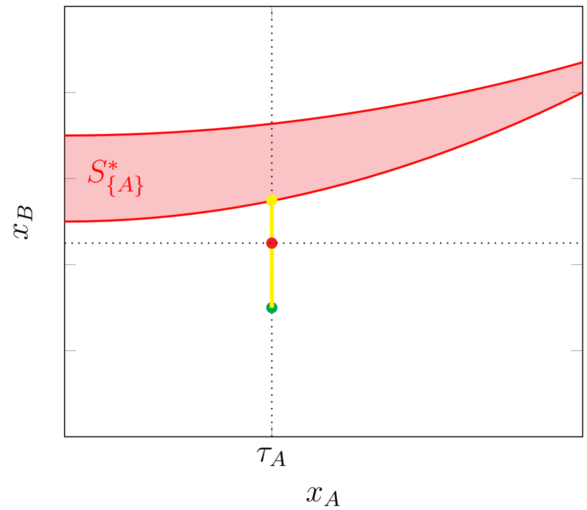


Figure 1: Second-stage policy

Intuitively, once a project is completed, the continuation problem resembles a single-project problem, where the payoff for completing the remaining project j is the marginal payoff derived from its completion, $q(\{A, B\}) - q(\{i\})$, and the distribution of completion times is adjusted to reflect the conditional distribution based on the information gained from the completion of the first project.

For each realization of τ_j, we denote by $v_i(x_i \mid \tau_j)$ the continuation value and stopping set $s_i^*(\tau_j)$ and a strict stopping set $s_i^{**}(\tau_j)$ according to the solution characterized in Section 3.1. In this way, we can define the optimal second stage stopping regions as

$$S^*_{\{i\}} := \{ \quad (x_A, x_B) \in \mathbb{R}^2_+ \ : \ x_{-i} \in s^*_{-i}(x_i) \quad \}$$

Figure 1 illustrates the second-stage stopping region $S^*_{\{A\}}$. In the figure, if project A was completed in the first stage at τ_A, the optimal second-stage policy would involve allocating attention to project B, moving vertically along the yellow line. The DM would either complete project B or stop upon reaching the red area, $S^*_{\{A\}}$.

First-stage policies

Before either project is completed, the DM must decide how to allocate a unit of attention between the two projects.

Definition 5. *A first-stage policy* $\alpha = (\alpha_A, \alpha_B)$ *consists of a pair of functions* α_i :

$\mathbb{R}_+ \to [0,1]$ *such that* $\alpha_A(t) + \alpha_B(t) \leqslant 1$.

As we argued in Section 2, we can, without loss of generality, focus on first-stage policies that *do not waste time*, i.e. policies α for which there exists a time $T^\alpha \in [0,\infty]$ such that $\alpha_A(t) + \alpha_B(t) = 1$ for $t \leqslant T^\alpha$ and $\alpha_A(t) + \alpha_B(t) = 0$ for $t > T^\alpha$.[4]

For a given first-stage policy α and time t, the cumulative attention allocated to project i by time t in the absence of any project completion is denoted by

$$x_i^\alpha(t) := \int_0^t \alpha_i(\tilde{t})\ d\tilde{t}.$$

We use $x^\alpha(t)$ to denote the vector $(x_A^\alpha(t), x_B^\alpha(t))$. Let $\mathcal{A}$ represent the set of all first-stage policies and let $X^\alpha := x^\alpha(T^\alpha)$ be the *first-stage stopping point* of first-stage policy α, i.e. the amount of attention that the DM allocates to each project before stopping, assuming no project is completed.

Given a first-stage policy $\alpha \in \mathcal{A}$, the set of completion times Ω can be partitioned in three regions, based on which project is completed first by a DM that follows first-stage policy α. Let $L^\alpha := \{x \in R_+^2 : x = x^\alpha(t), t \geqslant 0\}$ be the *path* of cumulative attention, i.e. the set of points in R_+^2 that can be reached over time by following policy α.

$$U_i^\alpha := \{\tau \in \Omega\ :\ \tau_i \geqslant x_i\ \&\ \tau_{-i} \leqslant x_{-i} \text{ for some } x \in L^\alpha\}$$

Figure 2 illustrates an arbitrary policy path L^α and the regions U_i^α for an arbitrary first stage policy $\alpha \in \mathcal{A}$. The DM that follows first-stage policy α completes project i in the first stage iff $\tau \in U_i^\alpha$. If $\tau > X^\alpha$, then τ is not in U_A^α or U_B^α and the DM does not complete any project before stopping in the first stage.

To find the optimal allocation policies, we focus on the ones that present optimal continuation in the second stage. Let σ^α be the policy that prescribes first-stage policy α in the first stage and optimal continuation in the second stage.

Proposition 2. *When projects are jointly continuous, the value* $V(\sigma^\alpha)$ *can be expressed as follows:*

$$V(\sigma^\alpha) = \int_0^{T^\alpha} G(x^\alpha(t)) \cdot \sum_{i=A,B} \alpha_i(t) \cdot \left[\ H_i(x^\alpha(t)) \cdot v_j(x_j^\alpha(t) \mid x_i^\alpha(t)) - c_i\ \right]\ dt \qquad (2)$$

[4] This is because there is no benefit for the DM in delaying the allocation of attention. This fact holds true even when the DM discounts future payoffs.

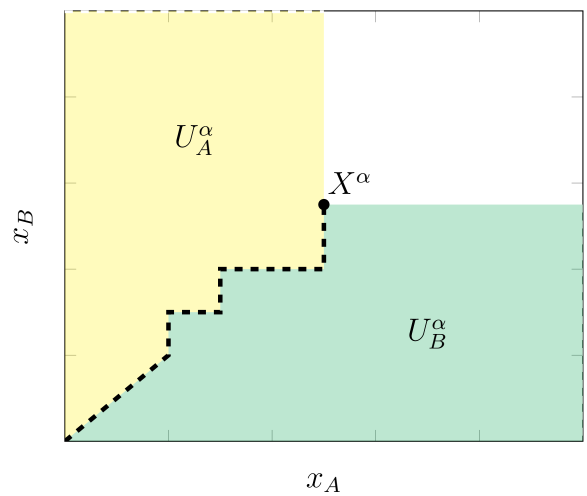


Figure 2: Sets U_A^α and U_B^α for a first stage policy α.

Where $H_i(x) := -\frac{\partial}{\partial x_i} \log G(x)$.

In Equation 2, $G(x^\alpha(t))$ represents the probability that the DM reaches time t while still in the first stage. If the DM is indeed in the first stage at time t, they allocate $\alpha_i(t)$ units of attention to project i. The term in square brackets represents the net expected flow payoff from allocating attention to project i during the first stage, as H_i captures the hazard rate for project i given that x_i attention has been allocated to the project, x_j attention was allocated to project j, and no project was completed so far.

4 Effective waste

To address the question of optimal policies, we first define an order over the space of first-stage policies. Then, we show that the expected payoff $V(\sigma^\alpha)$ of the DM is increasing in this order. By constructing this order, we are able to rule out suboptimal policies, thereby bringing us closer to identifying the optimal policy.

Definition 6. *A first-stage policy $\hat{\alpha}$ is* less wasteful *than α (denoted $\hat{\alpha} \succeq \alpha$) iff*

1. *α and $\hat{\alpha}$ share the same first-stage stopping point,i.e. $X^\alpha = X^{\hat{\alpha}}$.*

2. *$\left[U_i^{\hat{\alpha}} \setminus U_i^\alpha\right] \cap S^*_{\{-i\}} = \emptyset$ for $i = \{A, B\}$.*

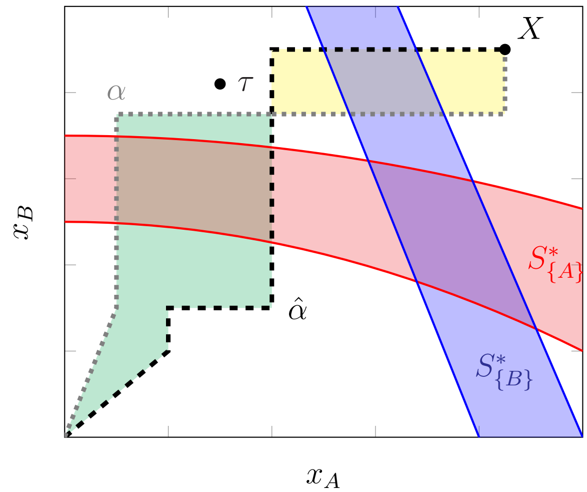


Figure 3: First-stage policy $\hat{\alpha}$ dominates first-stage policy α.

For a first-stage policy $\hat{\alpha}$ to dominate a policy α, two conditions must be met. First, both policies must have the same stopping point. Second, if there is a realization $\tau \in R^2_+$ such that $\hat{\alpha}$ completes project i and α completes project $-i$, then it must be that τ is not in $S^*_{\{-i\}}$.[5]

In Figure 3, we observe the paths of two first-stage policies, $\hat{\alpha}$ and α. Moreover, notice that the two policies have the same stopping point $X = X^\alpha = X^{\hat{\alpha}}$. The green area in between the curves, where $\hat{\alpha}$ allocates relative more attention to project A, does not intersect S^*_A. Likewise, the yellow area, where $\hat{\alpha}$ allocates relatively more to project B, does not intersect S^*_B. Thus, $\hat{\alpha}$ is less wasteful than α.

As it turns out, policies that are less wasteful yield a higher expected payoff than policy α, as formalized by the following result:

Theorem 1. *The expected payoff from using first-stage policies with optimal second-stage continuation is increasing in the effective waste partial order, i.e.*

$$\hat{\alpha} \succeq \alpha \quad \Rightarrow \quad V(\sigma^{\hat{\alpha}}) \geqslant V(\sigma^\alpha).$$

[5] Note that this definition uses the optimal second-stage policy given by the stopping regions $S^*_{\{A\}}$ and $S^*_{\{B\}}$. A similar definition can be constructed for arbitrary second-stage policies. With this alternative definition, it is possible to order policies with suboptimal second stage. Since, ultimately, we are interested in optimal policies—that involve optimality in the second stage—we refrain from this more general approach.

Proof. In Appendix B.1. □

The key idea behind the proof is to construct a potentially sub-optimal second-stage stopping policy such that, when $\hat{\alpha}$ is used in combination with this second-stage policy, the outcome is equivalent as using policy α with the optimal continuation in the second stage for every realization of completion times $\boldsymbol{\tau} \in \Omega$.[6] Note that Proposition 1 is vacuously true for perfect substitutes (where $q(\{A, B\}) = q(\{A\}) = q(\{B\})$) because in such cases, $S^*_A = S^*_B = \mathbb{R}^2_+$ and, therefore, there are no distinct policies $\hat{\alpha}$ and α such that $\hat{\alpha} \succeq \alpha$.

4.1 Waste-free policies

In this section, we focus on the subset of *waste-free* policies, which are policies that are less wasteful than any other policy. We argue that identifying the waste-free policy that yields maximal expected payoff can be obtained using standard optimization techniques, since the expected payoff of such policies is pinned down by their stopping point. In **??**, we present sufficient conditions under which the optimal first-stage policy is guaranteed to be waste-free.

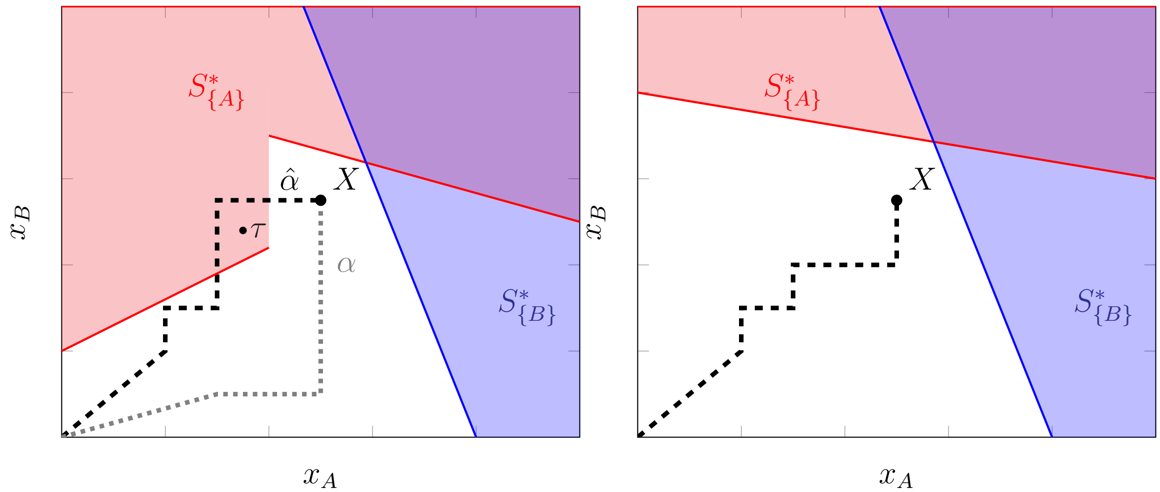
Definition 7. *An first-stage allocation policy α is* waste-free *if and only if for any $\tau \leqslant X^\alpha$ such that $\tau \in S^*_{\{i\}}$ it holds that that $\tau \in U^\alpha_i$, for $i = A, B$.*

Naturally, a waste-free policy is not dominated by any other policy. The waste-free property is illustrated in Figure 4. The dashed and dotted lines represent the paths of first-stage policies $\hat{\alpha}$ and α. Note that τ is lower than the stopping point X of policy $\hat{\alpha}$. Additionally, τ lies on $S^*_{\{A\}}$, but not on $U^{\hat{\alpha}}_A$. Thus, policy $\hat{\alpha}$ is not waste-free. Policy α, on the other hand, is waste-free. Any two policies that are waste-free and have the same stopping point induce expected payoff, as formalized in the following corollary.

Corollary 1. *If $\hat{\alpha}$ and α are waste-free and $X^\alpha = X^{\hat{\alpha}}$, then $V(\sigma^\alpha) = V(\sigma^{\hat{\alpha}})$.*

This corollary follows directly from Theorem 1: when two policies are waste-free and have the same stopping point, they are less wasteful than each other. Thus,

[6] Importantly, the same construction also applies to DMs who don't maximize expected payoffs. As long as the DM is only concerned with the distribution of payoffs, and not, for example, the order in which projects are completed, a similar construction works.



(a) Waste-free policies

(b) Encouraging projects.

Figure 4: The red and blue areas represent the optimal second-stage stopping points. The dashed and dotted lines represent the paths of different first-stage policies.

they must yield the same expected payoff. This reasoning implies that, rather than computing the value $V(\sigma^\alpha)$ for each waste-free first-stage policy α, it is sufficient to compute the value for each associated stopping point. This simplification is particularly useful when dealing with *encouraging* projects, as defined next.

Definition 8. *Projects are* encouraging *(*discouraging*) if c, q, and G are such that* $\inf s_i^*(\tau_j)$ *is decreasing (increasing) in τ_j for $i = A, B$.*

For any stopping point $X \in \mathbb{R}^2_+$ and project i, let σ_i^X be a policy with optimal stopping in the second stage, and first stage allocation given by α where $\alpha_i = 1$ for $t < X_i$ and $\alpha_i = 0$ for $t \in (X_i, X_i + X_j)$. When the projects are jointly continuous, we can apply Equation (2) to express $V(\sigma_i^X)$ as:

$$
\begin{aligned}
V(\sigma_i^X) = \int_0^{X_i} & G(\tilde{x}, 0) \cdot [\ H_i(\tilde{x}, 0) \cdot v_j(0|\tilde{x}) - c\ ]\ d\tilde{x} \\
& + \int_0^{X_j} G(X_i, \tilde{x}_j) \cdot [\ H_j(X_i, \tilde{x}_j) \cdot v_i(X_i|\tilde{x}_j) - c\ ]\ d\tilde{x}_j
\end{aligned}
\tag{3}
$$

Corollary 2. *If projects are encouraging, then $V(\sigma^\alpha) = V(\sigma_A^{X^\alpha}) = V(\sigma_B^{X^\alpha})$ for any waste-free policy α.*

Proof. This result relies on first noticing that when projects are encouraging, a policy α is waste-free if and only if $X_i^\alpha < S_i^*(X_{-i}^\alpha)$ for $i = A, B$. Then, applying Corollary 1, we obtain that the value of a first-stage policy α must be the same as the extreme policies, since these must also be waste-free. $\square$

The problem of determining the optimal stopping point, given a specific allocation order, $\max_{X \in \mathbb{R}_+^2} V(\sigma_i^X)$, is comparatively easier to solve than the more general problem of finding the overall optimal policy, $\max_{\alpha \in \mathcal{A}} V(\sigma^\alpha)$. This is because standard optimization techniques can be applied with Equation (3). While Theorem 1 allows us to rule out policies that are less wasteful than waste-free policies, it does not necessarily imply that the optimal policy must be waste-free.

For instance, consider the scenario where projects are perfect substitutes, i.e. $q(\{A\}) = q(\{B\}) = q(\{A, B\})$. In this case, there is no additional value to be gained in the second stage, making it optimal for the DM to stop immediately after the first project is completed. Thus, it is possible to show that the only policy that is waste free involves stopping allocating attention immediately at time zero. This, however, is not optimal for every possible parameters. In the next example, we obtain the optimal policy in the case of perfect substitutes.

Example 2. *Consider perfect substitutes, i.e.* $q(\{A\}) = q(\{B\}) = q(\{A, B\}) = Q$, *and let the projects be independent and with identically distributed completion times, i.e. there is a function* F *such that* $G(x_A, x_B) = (1 - F(x_A)) \cdot (1 - F(x_B))$. *Moreover, let the hazard rate of completion times* $h(x) := -F'(x)/(1 - F(x))$ *be strictly decreasing, with* $h(0) > c/Q$ *and* $\lim_{x \to \infty} h(x) < c/Q$. *Then,* $S_{\{A\}}^* = S_{\{B\}}^* = \mathbb{R}_+^2$ *and the unique optimal first stage policy is:*

$$\alpha_A(t) = \alpha_B(t) = \begin{cases} 0.5 & t < 2 \cdot h^{-1}(c/Q) \\ 0 & t > 2 \cdot h^{-1}(c/Q) \end{cases}$$

Intuitively, fixing the time that the DM is willing to work on the projects, the DM aims, for perfect substitutes, to minimize the expected time to complete one of them. With decreasing hazard rates, this involves always allocating attention to the project with the highest hazard rate. Since both projects have the same distribution, the optimal policy is to split attention equally betweem them. The DM stops when it is not profitable to continue, which occurs when the hazard rates reach c/Q.

In the next section, we examine conditions that ensure the optimal policy is waste-free.

5 Optimality of waste-free policies

In this section, we aim to identify conditions under which the optimal policy is waste-free. There are two aspects of the relationship between the projects: the impact of project completion on the final payoff—represented by the function q—and the information the DM obtains through resource allocation—specifically how much the DM learns about the completion time of project j by allocating attention to project i.

In this section, we analize the case where the informational channel is entirely shut down and provide the weakest condition on the payoff function that guarantees a frustration-free optimal policy. In Appendix D, we relax the information condition and establish a sufficient condition on information that, together with the condition on payoffs, guarantees a frustration-free optimal policy.

Definition 9. *The projects are* independent *if τ_A and τ_B are independent random variables, i.e. ther exists F_A and F_B such that*

$$S(x_A, x_B) = (1 - F_A(x_A))(1 - F_B(X_B))$$

For independent projects, there is no informational spillover between them: allocating attention to project A does not reveal any information about the completion time of project B, and vice versa. In the case of independent projects, the second-stage continuation value $v_i(x|\tau_j)$ and the stopping sets $s_i^*(\tau_j)$ are constant in the completion time τ_j. Thus, for independent projects, we can drop the dependency and denote $v_i(x)$ as the value of a single project problem with payoff $q(\{A, B\}) - q(\{-i\})$, cost c_i and distribution given by F_i and s_i^* the corresponding stopping set.

Note that a waste-free policy is one in which the stopping point is below the infimum of the stopping regions. Formally,

Remark 3. *For independent projects, σ^α is waste-free if and only if $X_i^\alpha \leqslant \inf s_i^*$ for $i = A, B$.*

Since there are no informational spillovers, the projects are related only through the payoff that the DM derives from completing them, which is determined by the function q. The complementarity between the projects is captured by the supermodularity of this function.

Definition 10. *The projects are* complements *if the function q is supermodular, that is,*

$$q(\{A\}) + q(\{B\}) \leqslant q(\{A, B\})$$

The projects are perfect complements *if $q(\{A\}) = q(\{B\}) = 0$.*

It turns out that complementarity and independence of the projects ensures a waste-free optimal policy, as is formalized in the following theorem.

Theorem 2. *If projects are complements and independent then there is an optimal policy that is waste-free.*

Proof. in Appendix C. □

To understand the intuition for this result, it is important to consider the incentives given by the marginal value of completing a project relative to stopping in each of the stages. In the case of perfect complements, the value of completing a project in the second-stage is $q(\{A, B\})$. The value obtained when completing project i in the first stage, instead, is $v_j(x_j)$, which is always lower. Thus, it is possible to show that the stopping region without any project completion, $S^*_\emptyset$, is a superset of $S^*_{\{A\}} \cup S^*_{\{B\}}$, which implies that the optimal policy must be waste-free.

However, this argument does not hold in the case of imperfect complements, as the marginal value of completing project i in the second stage relative to stopping, $q(\{A, B\}) - q(\{-i\})$, is not always greater than the value obtained by completing the same project in the first stage relative to stopping, $q(\{i\}) + v_j(x_j)$. Instead, the key to proving this result lies in showing that if the DM finds it optimal to allocate attention to project i in the first stage beyond the second-stage stopping region, then continuing to allocate attention to project j is also optimal.

Theorem 2 provides a sufficient condition to ensure a frustration-free optimal policy. Moreover, the supermodularity of q is the weakest condition on q that guarantees this: When q is not supermodular, it is possible to construct a survival rate function G such that the projects are independent and no optimal policy is frustration-free. Moreover, one can always construct G so that the hazard rate for each project is decreasing. This result is formalized in the following proposition.

Proposition 3. *If projects are not complements, there exist a survival function G such that the projects are independent, the hazard rate $H_i(x)$ of the projects is decreasing, and*

$$\max_{\alpha \in \mathcal{A}} V(\sigma^{\alpha}) > \max_{X \in \mathbb{R}^2_+} V(s_i^X) \qquad \textit{for } i = A, B$$

.

Proof. in Appendix C.2. □

6 Parametric application

In previous sections, we provide tools to analyze optimal resource allocation for interrelated projects in a general setup with very little structure in the distribution of completion times. In this section, we illustrate the applicability of these methods in a more cannonical, structured setting.

Consider two projects, each of which can be either *"difficult"* or *"easy"*, with hazard rates λ_L and λ_H, respectively. When the DM allocates attention α to project i, he completes the project at a rate $\alpha \cdot \lambda_i$, where λ_i is the hazard rate of the project. The difficulty of each project is randomly and independently determined, with project i having an ex-ante probability p_i of being easy.

This setup fits our previous framework, where the joint survival rate can be written as

$$G(x_A, x_B) \quad = \prod_{i=A,B} p_i e^{-\lambda_H x_i} + (1 - p_i) e^{-\lambda_L x_i}$$

Moreover, we assume that the projects are perfect complements, meaning that the reward of the DM is zero if only one project is completed. The DM receives a reward $q(\{A, B\}) = Q > 0$ only if both projects are completed.

6.1 Optimal allocation

If the reward Q is sufficiently high ($Q > 2 \cdot c/\lambda_L$), the DM completes both projects even if they are both known to be difficult. Thus, in this case, the DM never find it optimal to stop, regardless of uncertainty. On the other hand, if $Q < 2 \cdot c/\lambda_H$, the DM stops allocating attention immediately even if both projects were known to be easy. Thus, in that case, the DM would stop immediately even if there is uncertainty about the projects' difficulty. The more interesting case arises when the reward Q falls in between these two extremes, leading to the question of whether the DM would continue allocating attention if they knew that one if the projects was difficult and the other one easy.

Definition 11. *The projects are* cost-effective when assorted *when*

$$Q > \frac{c}{\lambda_H} + \frac{c}{\lambda_L}$$

When projects are cost-effective when assorted, it is optimal to complete both projects when it is known that one is difficult and the other one easy. Moreover, if the difficulty of the projects is known, the order in which the DM allocates the attention is irrelevant. However, with uncertainty, the order affects the final payoff of the DM because the flow of information about the difficulty of the projects is affected by how attention is allocated. Let i denote the *least promising* project and j the *most promising* one, meaning that the prior probabilities of the projects being easy are ordered as $p_i < p_j$.

The following proposition characterizes the optimal allocation of attention for the case of intermediate rewards.

Proposition 4. *Assume* $Q \in [2c/\lambda_H, 2c/\lambda_L]$. *Then,*

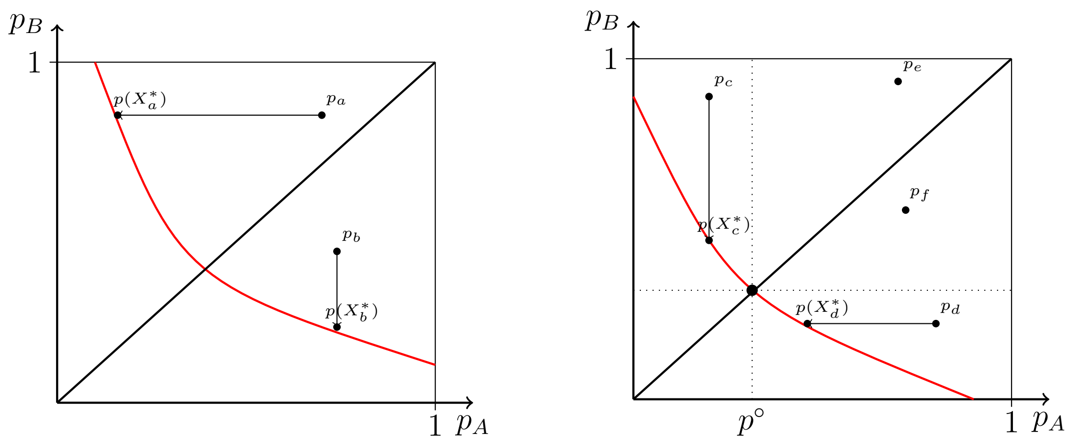
- *If the projects are not cost-effective when assorted, it is efficient to allocate attention only to the least promising project in the first stage. That is, for any optimal first-stage policy* α^*,

$$p_i < p_j \quad \Rightarrow \quad X_j^{\alpha^*} = 0.$$

- *If the projects are cost-effective when assorted, it is efficient to allocate more attention to the most promising project in the first stage. That is, for any optimal first-stage policy* α^*,

$$p_i < p_j \quad \Rightarrow \quad X_i^{\alpha^*} \leqslant X_j^{\alpha^*}.$$

Proof. in Appendix E. □



(a) Projects are cost-effective when assorted. (b) Projects are not cost-effective when assorted.

Figure 5: Projects in belief space.

The first part of Proposition 4 states that when the rewards are sufficiently low relative to the costs—i.e., when Q/c is lower than the expected completion time of both a difficult and an easy project—the optimal allocation of attention involves starting with the least promising project. This case is depicted in Figure 5a. In the figure, the horizontal axis represents the probability that project A is easy, while the vertical axis represents the same for project B. The red curve represents the boundry where the DM would decide to stop allocating attention to the projects. Note that the top-left and bottom-right corners lie below this boundry, which is consistent with the projects being not cost-effective when assorted.

If the initial beliefs are at the point p_a, the DM would optimally allocate attention to project A (the least promising one) first. As the DM continues allocating attention to project A without completing it, their beliefs become more pesimistic. The beliefs evolve along the arrow until they reach the boundary, at which point the DM stops. If, instead, the initial beliefs are at point p_b, the DM will start instead allocating attention to project B first, stopping when beliefs reach $p(X_b^*)$.

The second part of Proposition 4 states that when the rewards are sufficiently high relative to the costs—i.e., when Q/c is greater than the expected completion time of both a difficult and an easy project—the optimal allocation of attention involves giving more attention to the most promising project in the first stage. This case is depicted in Figure 5b.

In the figure, we label p° the belief at the intersection of the stopping boundary and the 45-degree line. When one of the initial beliefs is below p°, such as at points p_c and p_d, the DM optimally allocates attention to the most promising project before stopping at the boundary. If the initial belief for both projects is higher than p°, the DM allocates attention to both projects during the first stage until both projects reach the belief p°. This can only be achieved by allocating more attention to the most promising project.

7 Conclusion

In this paper, we examined the problem faced by a resource-constrained decision maker who allocates attention across multiple interrelated projects. The timing of these projects affects how information about their difficulty is obtained, which in turn shapes the optimal stopping decision.

We developed a framework for analyzing the problem of dynamic attention allocation and showed that—holding the cumulative attention allocated to each project constant, and conditional on no project completion—it is optimal to allocate attention in a way that minimizes effective waste, defined as situations where the DM wishes they had stopped allocating attention to a project earlier.

We showed how, in general, is possible to characterize optimal waste-free allocation policies. However, the overall optimal allocation policy may not always be waste-free. We then provided conditions for the optimal allocation policy to be waste-free. Specifically, we proved that when the difficulties of the projects is independent, the weakest condition that guarantees a waste-free optimal policy is complemetarity in payoffs.

We applied these findings to the case of perfect complementary projects with unknown constant completion rates. The solution involves prioritizing the least promising project when stakes are relatively low, and the most promising project when the stakes are high. This solution coincides with the fastest way to obtain information about an underlying decision-relevant state.

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A Omitted Proofs of Section 2

A.1 Proof of Proposition 1

Fixing an initial an state $x \in \mathbb{R}_+$. For any random variable with CDF H and support in $\mathbb{R}_+$ and parameters $\tilde{B}, \tilde{c} \in \mathbb{R}_+$, let

$$x^*(H, B, c) := \arg \max_{x' \geqslant x} \; [H(x') - H(x)] \cdot \tilde{B} - \tilde{c} \cdot \int_x^{x'} [1 - H(\tilde{x})] \, d\tilde{x}$$

Claim 1. *Let F and $\hat{F}$ be two absolutely continuous distributions such that F dominates $\hat{F}$ in the hazard rate order and $B, \hat{B}, c, \hat{c} \in \mathbb{R}_+$ such that $B/c \geqslant \hat{B}/\hat{c}$. Then*

$$x^*(F, B, c) \geqslant_{SSO} x^*(\hat{F}, \hat{B}, \hat{c}).$$

Where $\geqslant_{SSO}$ indicates strong set order.

Proof. Fix $x \in \mathbb{R}_+$ and define $\pi : \{0, 1\} \times [x, \infty) \to \mathbb{R}_+$ as follows:

$$\pi(1, x') := [F(x') - F(x)] \cdot B - c \cdot \int_x^{x'} [1 - F(\tilde{x})] \, d\tilde{x}$$

$$\pi(0, x') := [\hat{F}(x') - \hat{F}(x)] \cdot \hat{B} - \hat{c} \cdot \int_x^{x'} [1 - \hat{F}(\tilde{x})] \, d\tilde{x}$$

We will show that the functon π satisfies the Milgrom-Shannon single-crossing condition, meaning that for any $y, y' \in \mathbb{R}_+$ satisfying $y' > y \geqslant x$,

$$\pi(0, y') - \pi(0, y) \geqslant 0 \qquad \Rightarrow \qquad \pi(1, y') - \pi(1, y) \geqslant 0$$

First, note that any cdf H of an absolutely continuous random variable can be written as

$$H(y) = 1 - e^{-\int_{-\infty}^{y} h_H(\tilde{y})\ d\tilde{y}}$$

where $h_H(\tilde{y}) := H'(\tilde{y})/(1-H(\tilde{y}))$ represents the hazard rate. Thus, for any absolutely continuous random variable with cdf H and costants $\tilde{B}, \tilde{c} \in \mathbb{R}_+$

$$\begin{aligned}
& [H(y') - H(y)] \cdot \tilde{B} - \tilde{c} \cdot \int_y^{y'} [1 - H(\tilde{x})]\ d\tilde{x} && \geqslant 0 \\
\Leftrightarrow \quad & \frac{H(y') - H(y)}{1 - H(y)} \cdot \tilde{B} - \tilde{c} \cdot \int_y^{y'} \frac{1 - H(\tilde{x})}{1 - H(y)}\ d\tilde{x} && \geqslant 0 \\
\Leftrightarrow \quad & \left[1 - \frac{1 - H(y')}{1 - H(y)}\right] \cdot \tilde{B} - \tilde{c} \cdot \int_y^{y'} \frac{1 - H(\tilde{x})}{1 - H(y)}\ d\tilde{x} && \geqslant 0 \\
\Leftrightarrow \quad & \left[1 - e^{-\int_y^{y'} h_H(\tilde{y})\ d\tilde{y}}\right] \cdot \tilde{B} - \tilde{c} \cdot \int_y^{y'} e^{-\int_y^{\tilde{x}} h_H(\hat{x})\ d\hat{x}}\ d\tilde{x} && \geqslant 0
\end{aligned}$$

Thus, for any $y' > y \geqslant x$,

$$\begin{aligned}
\pi(0, y') - \pi(0, y) \geqslant 0 \quad \Leftrightarrow \quad & \left[1 - e^{-\int_y^{y'} h_{\tilde{F}}(\tilde{y})\ d\tilde{y}}\right] \cdot \tilde{B} - \tilde{c} \cdot \int_y^{y'} e^{-\int_y^{\tilde{x}} h_{\tilde{F}}(\hat{x})\ d\hat{x}}\ d\tilde{x} && \geqslant 0 \\
\Rightarrow \quad & \left[1 - e^{-\int_y^{y'} h_F(\tilde{y})\ d\tilde{y}}\right] \cdot \tilde{B} - \tilde{c} \cdot \int_y^{y'} e^{-\int_y^{\tilde{x}} h_F(\hat{x})\ d\hat{x}}\ d\tilde{x} && \geqslant 0 \\
\Rightarrow \quad & \left[1 - e^{-\int_y^{y'} h_F(\tilde{y})\ d\tilde{y}}\right] \cdot B - c \cdot \int_y^{y'} e^{-\int_y^{\tilde{x}} h_F(\hat{x})\ d\hat{x}}\ d\tilde{x} && \geqslant 0 \\
\Leftrightarrow \quad & \pi(1, y') - \pi(1, y) \geqslant 0
\end{aligned}$$

The first implication holds since $h_F(x) \geqslant h_{\tilde{F}}(x)$ for all $x \in \mathbb{R}_+$ and the second one because $B/c \geqslant \tilde{B}/\tilde{c}$ and $1 - e^{-\int_y^{y'} h_F(\tilde{y})\ d\tilde{y}} \geqslant 0$. Applying Theorem 4 in Milgrom and Shannon [1994], we obtain the desired result. □

Using the strong set order, we see that if $x \notin \hat{S}^*$, meaning that $x \notin x^*(\hat{F}, \hat{B}, \hat{c})$, it must be that $x \notin x^*(F, B, c)$ or $x \notin S^*$, which means that $S^* \subseteq \hat{S}^*$.

Claim 2. *Fix $B, c \in \mathbb{R}_+$ and a CDF H.*

$$y^*(x) := \quad \arg \quad \max_{y \geqslant x} \quad [H(y) - H(x)] \cdot B - c \cdot \int_x^y [1 - H(\tilde{x})] \; d\tilde{x}$$

is increasing in the strong set order.

Proof. Notice that the objective $[H(y) - H(x)] \cdot B - c \cdot \int_x^y [1 - H(\tilde{x})] \; d\tilde{x}$ satisfies increasing differences in (x, y), which implies that it satisfies Milgrom-Shannon Single Crossing Condition. Applying Theorem 4 from Milgrom and Shannon [1994], we obtain the desired result. $\square$

B Proofs of Section 4

B.1 Proof of Theorem 1

Proof. Consider first-stage policies $\hat{\alpha}, \alpha \in \mathcal{A}$ such that $\hat{\alpha} \succeq \alpha$. We will construct a suboptimal second-stage stopping regions $\hat{S}_{\{A\}}$ and $\hat{S}_{\{B\}}$ such that $\hat{\alpha}$ in combination with the suboptimal continuation yields the same payoff as the policy σ^α for every realization of completion times $\tau \in R_+^2$.

Let $\hat{S}_{\{A\}} = S^*_{\{A\}} \setminus (U_A^{\hat{\alpha}} \cap U_B^{\alpha})$ and $\hat{S}_{\{B\}} = S^*_{\{B\}} \setminus (U_B^{\hat{\alpha}} \cap U_A^{\alpha})$ and let $\hat{\sigma}$ be the policy that uses first-stage allocation according to $\hat{\alpha}$ and stops at $\hat{S}_{\{A\}}, \hat{S}_{\{B\}}$ in the second stage. We will show that, for every realization of τ, the policies $\hat{\sigma}$ and σ^α produce the same final payoff. Thus, $V(s^{\hat{\alpha}}) \geqslant V(\hat{\sigma}) = V(\sigma^\alpha)$, where the first inequality holds by optimality in the second stage.

Case 1: $\tau > X^\alpha$. In this case, policy σ^α completes no project before stopping. Since $\hat{\alpha} \succeq \alpha$, it must be that $X^{\hat{\alpha}} = X^\alpha > \tau$, thus $\hat{\sigma}$ also fails to complete a project in the first stage. The final payoff from policies σ^α and $\hat{\sigma}$ is therefore $-c_A \cdot X_A^\alpha - c_B \cdot X_B^\alpha$.

Case 2: $\tau \in U_A^\alpha$. In this case, σ^α completes project A in the first stage. Let

$$L_2^\alpha = \{x \in U_A^\alpha : x_A = \tau_A \quad \& \quad x_B < \tau_B\}$$

Note that σ^α completes B in the second stage if and only if $L_2^\alpha \cap S^*_{\{A\}} = \emptyset$. In this case, the final payoff for σ^α is $q(\{A, B\}) - c_A \cdot \tau_A - c_B \cdot \tau_B$. If $L_2^\alpha \cap S^*_{\{A\}} \neq \emptyset$, policy σ^α does not complete project B in the second stage and the final payoff is $q(\{A\}) - c_A \cdot \tau_A - c_B \cdot \underline{x}_B$ where $\underline{x}_B = \inf\{x_B : (\tau_A, x_B) \in L_2^\alpha \cap S^*_{\{A\}}\}$.

Case 2.1: $\tau \in U_A^\alpha \cap U_A^{\hat{\alpha}}$**.** In this case, policy $\hat{\sigma}$ also completes project A in the first stage. We will show that $L_2^\alpha \cap S^*_{\{A\}} = L_2^{\hat{\alpha}} \cap \hat{S}_{\{A\}}$.

Consider $x \in L_2^\alpha \cap S^*_{\{A\}}$. First, we show that $x \in U_A^{\hat{\alpha}}$. Towards a contradiction, assume that $x \notin U_A^{\hat{\alpha}}$. Then, because $x_A = \tau_A < X_A^{\hat{\alpha}}$, it must be that $x \in U_B^{\hat{\alpha}}$. Therefore, $x \in U_A^\alpha \cap U_B^{\hat{\alpha}}$, which implies that $x \notin S^*_{\{A\}}$ since $\hat{\alpha} \succeq \alpha$, which completes the contradiction. Because $x_A = \tau_A$, $x_B < \tau_B$ and $x \in U_A^{\hat{\alpha}}$, then $x \in L_2^{\hat{\alpha}}$.

Next, we show that $x \in \hat{S}_{\{A\}}$. For this, we already by assumption have that $x \in S^*_{\{A\}}$. It remains to show that $x \notin U_A^{\hat{\alpha}} \cap U_B^\alpha$, which holds since $x \in U_A^\alpha$ and therefore $x \notin U_B^\alpha$.

Thus, we proved so far that $L_2^\alpha \cap S^*_{\{A\}} \subseteq L_2^{\hat{\alpha}} \cap \hat{S}_{\{A\}}$. For the other direction, consider $x \in L_2^{\hat{\alpha}} \cap \hat{S}_{\{A\}}$. First, we show that $x \in U_A^\alpha$. Suppose, towards a contradiction, that $x \notin U_A^\alpha$. Then, because $x_A = \tau_A < X_A^{\hat{\alpha}}$, it must be that $x \in U_B^\alpha$.

Next, we prove that $x \in S^*_{\{A\}}$. This is an immediate consequence of $x \in \hat{S}_{\{A\}} \subseteq S^*_{\{A\}}$. Thus, we proved that $L_2^{\hat{\alpha}} \cap \hat{S}_{\{A\}} \subseteq L_2^\alpha \cap S^*_{\{A\}}$ which, together with the previous result, proves the equality.

Thus, the final payoff is the same for $\hat{\sigma}$ and σ^α:

$$\begin{aligned} q(\{A, B\}) - c_A \cdot \tau_A - c_B \cdot \tau_B &\qquad \text{when} \qquad L_2^\alpha \cap S^*_{\{A\}} = \emptyset \\ q(\{A\}) - c_A \cdot \tau_A - c_B \cdot \underline{x}_B &\qquad \text{when} \qquad L_2^\alpha \cap S^*_{\{A\}} \neq \emptyset \end{aligned}$$

Case 2.2: $\tau \in U_A^\alpha \cap U_B^{\hat{\alpha}}$**.** In this case, $\hat{\sigma}$ completes project B first. We will show that $\hat{\sigma}$ must complete project A in the second stage and s^α completes project B in the second stage. Thus, both policies complete both projects, and their final payoff is $q(\{A, B\}) - c_A \cdot \tau_A - c_B \cdot \tau_B$.

Suppose by contradiction that $\hat{\sigma}$ doesn't complete project A in the second stage. This means that there exists an $x_A < \tau_A$ such that $(x_A, \tau_B) \in U_B^{\hat{\alpha}} \cap \hat{S}_{\{B\}}$. However, $\tau \in U_A^\alpha$ implies that (x_A, τ_B) must be in $U_{\{A\}}^\alpha$ as well. Therefore, $x \in U_A^\alpha \cap U_B^{\hat{\alpha}}$, which means that $x \notin \hat{S}_{\{B\}}$, a contradiction.

Case 3 $\tau \in U_B^\alpha$**.** This case is symmetric to Case 2.

With these three cases, we show that the payoff yield by σ^α and $\hat{\sigma}$ is the same with probability one, which means that $V(\sigma^\alpha) = V(\hat{\sigma})$, finishi □

C Proof of Theorem 2

To prove Theorem 2, we proceed as follows. We begin by assuming the existence of an optimal first-stage policy α. The first part of the proof, Proposition 5, consist on obtaining an upper bound on the second-stage value of the projects at the first-stage stopping point of α. Next, we examine the last point at which an optimal first-stage policy would cross a stopping region, and show that the existence of such point generates a contradiction. We do so by apply Proposition 1 to identify other first-stage policies that dominate α and hence must also be optimal. This proves that an optimal first-stage policy with a finite stopping point cannot cross a stopping region, and therefore is frustration-free.

Proposition 5. *Assume that projects are independent and complements. Let α be an optimal first-stage policy. Then,*

$$v_i(X^\alpha) \leqslant q(\{A,B\}) - q(\{A\}) - q(\{B\}) \qquad \textit{for} \;\; i = A, B$$

Proof. Let s_i^* be the stopping set of the single project problem with payoff $q(\{A,B\}) - q\{-i\}$, cost c_i, and distribution given by cdf F_i and let $\bar{x}_i$ be the point at which the DM optimally stops if he completes the project j exactly at X^α, i.e. $\bar{x}_i := \inf\ [X_i^\alpha, \infty) \cap s_i^*$.[7] We can write the second-stage continuation payoff of the DM exactly at X^α as

$$P \cdot [q(\{A,B\}) - q(\{j\})] - K \tag{4}$$

Where $P := \Pr(\tau_i < \bar{x}_i \mid \tau_i > X_i^\alpha)$ is the conditional probability that the DM completes project i in the second stage and $K := c_i \cdot \mathbb{E}[\min\{\tau_i, \bar{x}_i\} - X_i^\alpha \mid \tau_i > X_i^\alpha\}]$ the second stage expected cost.

Assume, towards a contradiction, that $v_i(X^\alpha) > q(\{A,B\}) - q(\{A\}) - q(\{B\})$.

[7] with the usual convention that the infimum is infinite if for an empty set.

Then, using Equation (4),

$$P \cdot [q(\{A,B\}) - q(\{j\})] - K > q(\{i,j\}) - q(\{i\}) - q(\{j\})$$

Adding and subtracting $P \cdot q(\{i\})$ on the left-hand-side, we get

$$P \cdot [q(\{i,j\}) - q(\{j\}) - q(\{i\})] + P \cdot q(\{i\}) - K > q(\{i,j\}) - q(\{i\}) - q(\{j\})$$

Thus, by rearranging, we get

$$P \cdot q(\{i\}) - K > [1-P] \cdot [q(\{A,B\}) - q(\{A\}) - q(\{B\})] \geqslant 0$$

Where the second inequality holds since projects are complements. Note that the strict inequality implies that P is positive, which means that $\bar{x}_i > X_i^\alpha$.

$P \cdot q(\{i\}) - K > 0$ contradicts the optimality of policy σ^α: consider the DM that follows policy σ^α and reaches X^α in the first stage, i.e. without completing any of the projects. At this point the policy σ^α would indicate him to stop. Instead, the DM can do better by disregarding project j and continuing allocating attention to project i until completing it or hitting $\bar{x}_i$, what would have a positive expected continuation value $P \cdot q(\{i\}) - K$. □

Theorem 2 If projects are complements and independent, then there is an optimal first-stage strategy that is waste-free.

Proof. Let $\underline{s}_i := \inf(s_i^{**})$. We will show that any optimal first-stage policy α must satisfy $X_i^\alpha \leqslant \underline{s}_i$. This, combined with the independence assumption and Corollary 2, implies that any optimal policy is waste-free.

Let α be an optimal first-stage policy. If $X_i^\alpha \leqslant \underline{s}_i$ for $i = A, B$, then α is waste-free and the proof is complete.

Suppose instead that $X_i^\alpha > \underline{s}_i$ for at least one of the projects. Let $\hat{t}$ be the last time at which the path of α is in one of the stopping regions, i.e.

$$\hat{t} \quad = \quad \sup \ \{ \ t \in \mathbb{R} \ : \ x^\alpha(t) \in S_A^{**} \cup S_B^{**} \ \}$$

Note that $X_i^\alpha > \underline{s}_i \geqslant 0$. This, together with the fact that S_i^{**} is open, implies that $\hat{t}$ must be strictly greater than zero. Moreover, since S_i^{**} a finite union of open sets, one of the following cases must hold, depending on which stopping region the first-stage policy α crosses last.

Case i. First-stage policy α crosses stopping region of project B last, i.e. there exists an $\bar{\epsilon} > 0$ such that $x^{\alpha}(\hat{t} - \epsilon) \in S_B \cap S_A^c$ for all $\epsilon \in (0, \bar{\epsilon})$.

Case ii. First-stage policy α crosses stopping region of project B last, i.e. there exists an $\bar{\epsilon} > 0$ such that $x^{\alpha}(\hat{t} - \epsilon) \in S_A \cap S_B^c$ for all $\epsilon \in (0, \bar{\epsilon})$

Case iii. First-stage policy α crosses both stopping regions last, i.e. there exists an $\bar{\epsilon} > 0$ such that $x^{\alpha}(\hat{t} - \epsilon) \in S_A \cap S_B$ for all $\epsilon \in (0, \bar{\epsilon})$.

We will show that for each of these cases we can arrive to a contradiction, and thus it cannot be the case that $X_i^{\alpha} > \underline{s}_i$ for neither of the projects.

- **Case (i)** [case *(ii)* is symmetric.]

  Take some $\hat{\epsilon} \in (0, \bar{\epsilon})$. We define the first-stage policy $\hat{\alpha}$ that has the same stopping point as α, is equivalent to α before $\hat{t} - \hat{\epsilon}$ and then allocates attention first to project A. Formally, let $\tilde{t} := \hat{t} - \hat{\epsilon} + X_A^{\alpha} - x_A^{\alpha}(\hat{t} - \hat{\epsilon})$.

$$\hat{\alpha} = \begin{cases} \alpha & t < \hat{t} - \hat{\epsilon} \\ (1,0) & t \in (\hat{t} - \hat{\epsilon}, \tilde{t}) \\ (0,1) & t \in (\tilde{t}, T^{\alpha}) \\ (0,0) & t > T^{\alpha}. \end{cases}$$

  The first-stage policy $\hat{\alpha}$ dominates α, so by Theorem 1, it must be that $\hat{\alpha}$ is also optimal. Since $\hat{\alpha}$ is optimal and allocates exclusive attention to project B after $\tilde{t}$, it must be, by Lemma 2, that

$$\Pr[\tau_B \leqslant X_B^{\alpha} | \tau_B > x_B^{\hat{\alpha}}(\tilde{t})] \cdot [q(B) + v_A(X^{\alpha})] - c_B \cdot \mathbb{E}(\min\{\tau_B, X_B^{\alpha}\} - x_B^{\hat{\alpha}}(\tilde{t})) \geqslant 0$$

  Note that this implies that $\Pr[\tau_B \leqslant X_B^{\alpha} | \tau_B > x_B^{\hat{\alpha}}(\tilde{t})] \geqslant 0$. Also, by Proposition 5, $v_A(X^{\alpha}) + q(\{B\}) \leqslant q(\{A, B\}) - q(\{A\})$. Thus,

$$\Pr[\tau_B \leqslant X_B^{\alpha} | \tau_B > x_B^{\hat{\alpha}}(\tilde{t})] \cdot [q(\{A, B\}) - q(\{A\})] - c_B \cdot \mathbb{E}(\min\{\tau_B, X_B^{\alpha}\} - x_B^{\hat{\alpha}}(\tilde{t})) \geqslant 0$$

  This contradicts the fact that $x_B^{\hat{\alpha}}(\tilde{t}) \in s_B^{**}$.

- **Case** (*iii*)**.** We consider three separate subcases, depending on how the stopping point relates to the point at which the first-stage policy α crosses a stopping region last.

  Case (iii.a.) Stopping point X^α is far from both stopping regions: $x_i^\alpha(\hat{t}) < X_i^\alpha$ for $i = A, B$.

  Case (iii.b.) Stopping point X^α is far from region A, but close to stopping region B: $x_A^\alpha(\hat{t}) < X_A^\alpha$ and $x_B^\alpha(\hat{t}) = X_B^\alpha$. [The other case is symmetric.]

  Case (iii.c.) Stopping point X^α is in both stopping regions: $\hat{t} = T^\alpha$.

- **Case (iii.a.)**. In this case we generate a contradiction by constructing two first-stage policies that have the same stopping point as α but prioritize one of the projects after time $\hat{t}$. Formally, we define α_1 and α_2 as follows:

$$\alpha_1 = \begin{cases} \alpha & t < \hat{t} \\ (1,0) & t \in (\hat{t}, \hat{t} + X_A^\alpha - x_A^\alpha(\hat{t})) \\ (0,1) & t \in (\hat{t} + X_A^\alpha - x_A^\alpha(\hat{t}), T^\alpha) \\ (0,0) & t > T^\alpha. \end{cases} \qquad \alpha_2 = \begin{cases} \alpha & t < \hat{t} \\ (0,1) & t \in (\hat{t}, \hat{t} + X_B^\alpha - x_B^\alpha(\hat{t})) \\ (1,0) & t \in (\hat{t} + X_B^\alpha - x_B^\alpha(\hat{t}), T^\alpha) \\ (0,0) & t > T^\alpha. \end{cases}$$

  Both α_1 and α_2 dominate α, and therefore must also be optimal. Since α_1 is optimal, it must be that

$$\Pr[\tau_B \leqslant X_B^\alpha \mid \tau_B > x_B^\alpha(\hat{t})] \cdot [q(\{A,B\}) - q(\{A\})] - c_B \cdot \mathbb{E}\left(\min\{\tau_B, X_B^\alpha\} - x_B^\alpha(\hat{t}) \mid \tau_B > x_B^\alpha(\hat{t})\right) \geqslant 0$$

  Moreover, since $X_B^\alpha > x_B^\alpha(\hat{t})$, it must be that $\Pr[\tau_B \leqslant X_B^\alpha \mid \tau_B > x_B^\alpha(\hat{t})] > 0$.

$$v_A(X^\alpha) + q(\{B\}) \geqslant q(\{A,B\}) - q(\{A\}) \tag{5}$$

  Next, we show that $v_B(X^\alpha) = 0$. Suppose towards a contradiction that $v_B(X^\alpha) > 0$. Then, there exists $\hat{x} > X_B^\alpha$ such that

$$\Pr[\tau_B \leqslant \hat{x} \mid \tau_B > X_B^\alpha] \cdot [q(\{A,B\}) - q(\{A\})] - c_B \cdot \mathbb{E}\left(\min\{\tau_B, \hat{x}\} - X_B^\alpha \mid \tau_B > X_B^\alpha\right) > 0$$

  Thus, by Equation (5),

$$\Pr[\tau_B \leqslant \hat{x} \mid \tau_B > X_B^\alpha] \cdot [q(\{B\}) + v_A(X^\alpha)] - c_B \cdot \mathbb{E}\left(\min\{\tau_B, \hat{x}\} - X_B^\alpha \mid \tau_B > X_B^\alpha\right) > 0$$

Which contradicts the optimality of stopping at X^α, and therefore proves that $v_B(X^\alpha) = 0$. We will use this together with the optimality of α_2 and supermodularity of q to arrive to a contradiction.

Since α_2 is optimal, we apply Lemma 2 to get that

$$\Pr[\tau_A \leqslant X_A^\alpha \mid \tau_A > x_A^\alpha(\hat{t})]\cdot[q(\{A\})+\underbrace{v_B(X^\alpha)}_{=0}]-c_A\cdot\mathbb{E}\left(\min\{\tau_A, X_A^\alpha\} - x_A^\alpha(\hat{t}) \mid \tau_A > x_A^\alpha(\hat{t})\right) \geqslant 0$$

$X_A^\alpha > x_A^\alpha(\hat{t})$ implies that $\Pr[\tau_A \leqslant X_A^\alpha \mid \tau_A > x_A^\alpha(\hat{t})] > 0$. Thus, by strict supermodularity of q, the previous equation implies

$$\Pr[\tau_A \leqslant X_A^\alpha \mid \tau_A > x_A^\alpha(\hat{t})]\cdot[q(\{A,B\})-q(\{B\})]-c_A\cdot\mathbb{E}\left(\min\{\tau_A, X_A^\alpha\} - x_A^\alpha(\hat{t}) \mid \tau_A > x_A^\alpha(\hat{t})\right) > 0$$

This contradicts the fact that $x_A^\alpha(\hat{t})$ is in the closure of s_A^{**}, which is a subset of s_A^*.

- **Case (iii.b.)** In this case, the stopping point X^α is in the closure of to the strict stopping region of project A. This implies that $X_A^\alpha \in s_A^*$, thus $V_A(X^\alpha) = 0$. Note that this case is only possible if α allocates exclusively to project B after $\hat{t}$. By this and optimality of α, we can apply Lemma 2 to get that

$$\Pr[\tau_B \leqslant X_B^\alpha \mid \tau_B > x_B^\alpha(\hat{t})]\cdot[q(\{B\})+\underbrace{v_A(X^\alpha)}_{=0}]-c_B\cdot\mathbb{E}\left(\min\{\tau_B, X_B^\alpha\} - x_B^\alpha(\hat{t}) \mid \tau_B > x_B^\alpha(\hat{t})\right) \geqslant 0$$

Since $X_B^\alpha > x_B^\alpha(\hat{t})$ implies that $\Pr[\tau_B \leqslant X_B^\alpha \mid \tau_B > x_B^\alpha(\hat{t})]$ is strictly positive, we can obtain, by applying the strict sumpermodularity of q in the previous equation,

$$\Pr[\tau_B \leqslant X_B^\alpha \mid \tau_B > x_B^\alpha(\hat{t})]\cdot[q(\{A,B\})-q(\{A\})]-c_B\cdot\mathbb{E}\left(\min\{\tau_B, X_B^\alpha\} - x_B^\alpha(\hat{t}) \mid \tau_B > x_B^\alpha(\hat{t})\right) > 0$$

this contradicts the fact that $x_A^\alpha(\hat{t})$ is in the closure of s_A^{**} and therefore in s_A^*.

- **Case (iii.c.)** In this case, the stopping point of α is at both stopping regions. Therefore, it is not possible to proceed as before by constructing first-stage policies that dominate α and end up allocating exclusively to one of the projects. Instead, the strategy for the proof lies in realizing that at the last instant in the first stage the continuation payoff is negative, which contradicts the possibility of α being optimal.

The continuation value of following the policy s^α at time $\hat{t} - \hat{\epsilon}$ can be written as:

$$V := \int_{T^\alpha - \hat{\epsilon}}^{T^\alpha} \frac{G(x^\alpha(\tilde{t}))}{G(x^\alpha(T^\alpha - \hat{\epsilon}))} \cdot \sum_{k=A,B} \alpha_k (h_k(\hat{t}) \cdot q(\{k\}) - c_k)\ d\tilde{t}$$

By strict supermodularity of q,

$$V < \int_{T^\alpha - \hat{\epsilon}}^{T^\alpha} \frac{G(x^\alpha(\tilde{t}))}{G(x^\alpha(T^\alpha - \hat{\epsilon}))} \cdot \sum_{k=A,B} \alpha_k (h_k(\hat{t}) \cdot [q(\{A,B\}) - q(\{A,B\} \setminus \{k\})] - c_k)\ d\tilde{t}$$

$(h_B \cdot [q(\{A,B\}) - q(\{A\})] - c_B)]$ and $(h_A \cdot [q(\{A,B\}) - q(\{B\})] - c_A)]$ are negative for all $t \in (T^\alpha - \hat{\epsilon}, T^\alpha)$. Thus, $V < 0$. This contradicts the optimality of α.

□

C.1 Lemmata

Lemma 1. *For a one-project problem (F, Π, c) with F absolutely continuous the set of strict stopping points s^{**} is right-open, i.e. if $x \in s^{**}$, there exists an $\bar{\epsilon} > 0$ such that $x + \epsilon \in s^{**}$ for all $\epsilon \in [0, \bar{\epsilon})$.*

Proof. Let $x \in s^{**}$. Then $h(\hat{x}) := f(\hat{x})/1 - F(\hat{x})$ must be weakly lower than c/B for all $\hat{x}$ in a right-neighborhood U of x. This implies that the function $v(\cdot)$ is weakly increasing in U.

- Suppose $v(\hat{x}) = 0$ for all $\hat{x} \in U$. Consider $x_0 \in U$. For every $x_1 > x_0$, let x' be in $U \cap (x_0, x_1)$. Then $\pi(x_1, x_0) \leqslant \int_{x_0}^{x'} S(t) \cdot [h(t) \cdot B - c]\ d\tilde{t} + S(x') \cdot v(x') < 0$.
- Suppose that the value is strictly positive for part U and let $\bar{\epsilon} = \inf\{e \in U : v(x+e) > 0\}$. Then $\bar{\epsilon} > 0$. To prove it, assume by contradiction that $\bar{\epsilon} = 0$, then there is a $x' > \sup U$ such that $\pi(x', \hat{x}) > 0$ for all $\hat{x} \in U$. Taking limit $\hat{x} \to x$, we obtain by continuity of π, that $\pi(x', x) \geqslant 0$, which contradicts $x \in s^{**}$. Finally, for any $\epsilon \in (0, \bar{\epsilon})$, $\pi(x', x + \epsilon)$ is negative for all $x' > x + \epsilon$.

□

Lemma 2. *Let projects be independent and s^α be an optimal policy. Let $\underline{t}$ be such that $\alpha_i(t) = 0$ for all $t > \underline{t}$ and define $\underline{x} := x_j^\alpha(\underline{t})$. Then,*

$$\int_{\underline{x}}^{X_j^\alpha} \frac{1 - F_j(\tilde{x})}{1 - F_j(\underline{x})} \cdot [\ h_j(\tilde{x}) \cdot [q(\{j\}) + v_i(X^\alpha)] - c_j\ ]\ d\tilde{x} \geqslant 0$$

Proof. Since payoffs are time-separable, an optimal policy must induce a positive continuation value at each point. In general, the continuation value of a policy s^α at time t_0 in the first stage is:

$$\int_{t_0}^{T^\alpha} \frac{G(x^\alpha(t))}{G(x^\alpha(t_0))} \sum_{k=A,B} \alpha_k(t) \cdot \left[\ H_k(x^\alpha(t)) \cdot v_{-k}(x_{-k}^\alpha(t) \mid x_k^\alpha(t)) - c_k\ \right]\ dt$$

Projects are independent, hence the hazard rate in the first stage is the same as the hazard rate in the second stage $H_j(x^\alpha(t)) = h_j(x_j^\alpha(t))$. Also, independence implies that the continuation value is constant and equal to $v_i(X^\alpha)$. Changing variables, we obtain the desired result. $\square$

C.2 Proof of Proposition 3

Proof. Since q is not supermodular, $q(\{A\}) + q(\{B\}) > q(\{A, B\})$. Thus,

$$\frac{c_A}{q(\{A\})} < \frac{c_A}{q(\{A, B\}) - q(\{B\})}$$

Let i be the project with highest $c_i(q(A, B) - q(\{i\}))$.

We construct the hazard rate of each project be such that: (i) h_i is strictly decreasing. (ii) $h_i(0) = \frac{c_i}{q(\{A,B\}) - q(j)}$. (iii) $\lim_{x\to\infty} h_i(x) = \frac{c_i}{q(\{i\})}$.

Given that $h_i < \frac{c_i}{q(\{A,B\}) - q(j)}$, the DM stops after the first success: at most one project will be developed. Given that $h_i > \frac{c_i}{q(\{i\})}$, it is never optimal to stop before the first success. Thus, exactly one project will be developed.

Thus, the optimal first-stage strategy of the DM is the same as it would be in the case of perfect substitutes: it is optimal for the agent to always work on the project with the highest flow payoff, i.e. in project i iff

$$h_i \cdot q(\{i\}) - c_i > h_j \cdot q(\{j\}) - c_j$$

Moreover, the DM starts with project i. However, after some time with no success, the agent would like to switch to project j since

$$\lim_{x\to\infty} h_i \cdot q(\{i\}) - c_i = 0 < h_j(0) \cdot q(\{j\}) - c_j$$

□

D Affiliated projects

In Section 5, we showed that for independent projects, complementarity is the minimum requirement to guarantee an optimal policy that is frustration-free. In this section, we expand on this result by relaxing the independence assumption by examining the situation where the projects are possitively affiliated.

Definition 12. *The projects are* positively (negatively) affiliated *if they are jointly continuous and the associated density function* $f : \mathbb{R}_+^2 \to \mathbb{R}$ *is log-supermodular (log-submodular), i.e.*

$$f(\tau \vee \tau') \cdot f(\tau \wedge \tau') \geqslant (\leqslant) f(\tau) \cdot f(\tau').$$

We will provide a sufficient condition for the optimal strategy to be frustration-free under this relaxed assumption. A first thing to notice is how affiliation shapes the optimal stopping decision in the second stage.

Lemma 3. *Positive (negative) affiliated projects are encouraging (discouraging).*

Proof. Let X and Y be two jointly continuous random variables. Let $f : \mathbb{R}^2 \to \mathbb{R}_{++}$ be their *joint probability density function* and

$$f_X(x) := \int_{-\infty}^{\infty} f(x,y)\ dy \qquad \text{and} \qquad f_Y(y) := \int_{-\infty}^{\infty} f(x,y)\ dx$$

be their respective *marginal density functions.* Finally, let

$$f_{Y|X}(y|x) := \frac{f(x,y)}{f_Y(y)} \quad \text{and} \quad f_{X|Y}(x|y) := \frac{f(x,y)}{f_X(x)}$$

be the *conditional density functions.*

Lemma 4. *If f is log-supermodular (log-submodular) then, for any $x' \geqslant x$, the conditional random variable $Y|X = x$ dominates $Y|X = x'$ in the hazard rate order.*

Proof. Let f be log-supermodular. We want to see that

$$h_{Y|X}(y|x) := \frac{f_{Y|X}(y|x)}{1 - F_{Y|X}(y|x)}$$

is decreasing in x for all y. For $x' \geqslant x$ and $y' \geqslant y$,

$$\begin{aligned}
& f(x', y) \cdot f(x, y') \leqslant f(x, y) \cdot f(x', y') \\
\Rightarrow \quad & f_X(x') \cdot f_{Y|X}(y|x') \cdot f_X(x) \cdot f_{Y|X}(y'|x) \leqslant f_X(x) \cdot f_{Y|X}(y|x) \cdot f_X(x') \cdot f_{Y|X}(y'|x') \\
\Rightarrow \quad & f_{Y|X}(y|x') \cdot f_{Y|X}(y'|x) \leqslant f_{Y|X}(y|x) \cdot f_{Y|X}(y'|x') \\
\Rightarrow \quad & \int_y^\infty f_{Y|X}(y|x') \cdot f_{Y|X}(y'|x)\ dy' \leqslant \int_y^\infty f_{Y|X}(y|x) \cdot f_{Y|X}(y'|x')\ dy' \\
\Rightarrow \quad & f_{Y|X}(y|x') \cdot [1 - F_{Y|X}(y|x)] \leqslant f_{Y|X}(y|x) \cdot [1 - F_{Y|X}(y|x')] \\
\Rightarrow \quad & \frac{f_{Y|X}(y|x')}{1 - F_{Y|X}(y|x')} \leqslant \frac{f_{Y|X}(y|x)}{1 - F_{Y|X}(y|x)}
\end{aligned}$$

(The proof for the case of f sub-modular is similar but with all inequalities reversed.) ∎

Sketch of the proof. Affiliation implies that the conditional hazard rate is ordered: when projects are positively affiliated, the second stage hazard rate $h_i(x|\tau_j)$ is increasing in τ_j for all i (decreasing for negatively affiliated projects). This hazard rate dominance implies that $S_j(\tau) \subseteq S_j(\tau')$ for any $\tau' \leqslant \tau$, which immediately implies monotonicity of $\inf\{S_j(\cdot)\}$.

Let $\tau_j'' \geqslant \tau_j'$. When the projects are negatively affiliated, by Lemma 4, $\tau_i|\tau_j''$ hazard rate dominates $\tau_i|\tau_j'$. For any τ_j, let $F_{\tau_i|\tau_j}$ be the conditional cumulative distribution. The stopping problem at the second stage can be written as:

$$\max_{x' \geqslant x} \qquad [F_{\tau_i|\tau_j}(x') - F_{\tau_i|\tau_j}(x)] \cdot [q(\{i,j\}) - q(\{j\})] - c \cdot \int_x^{x'} [1 - F_{\tau_i|\tau_j}(\tilde{x})]\ d\tilde{x}.$$

We can apply Claim 1. This implies that the solution to the problem with $\tau_j = \tau_j''$ is larger in the strong set order than the solution to the problem with $\tau_j = \tau_j'$. Thus, $\bar{x}_i(\tau_j'') \leqslant \bar{x}_i(\tau_j')$. ∎

∎

When projects are positively affiliated, an increase in τ_j is an indicator of higher increased attention required for project i, meaning that to complete project i with after a certain amoint of cummulative attention decreases. As a result, if the DM is willing to stop project i at x after τ_j, then they would also prefer to stop at x for any higher value $\hat{\tau}_j > \tau_j$.

Notice that, by Corollary 2, when projects are positively affiliated, a first-stage policy α is frustration-free if and only if $X_A^\alpha \leqslant \inf\{S_A(X_B^\alpha)\}$ and $X_B^\alpha \leqslant \inf\{S_B(X_A^\alpha)\}$. This means that, for positively affiliated projects, the value V of frustration-free policies is pinned-down by X^α. Moreover, to compute the value of any regret-free policy, we can simply assume that attention is first allocated to any of the projects.

Finally, sometimes projects are such that once stopping is optimal, it is optimal forever. When this is true for a project in the second stage, we say that the project satisfies the *threshold property.* Formally,

Definition 13. *Project i satisfies the* threshold property *iff $S_i(\tau_j)$ is convex and unbounded for all τ_j.*

A sufficient condition for a project to satisfy the threshold property is to have a decreasing conditional hazard rate. This condition is extensively assumed in experimentation setups. Next, we present the main theorem regarding affiliated projects.

Theorem 3. *If projects are complements, negatively affiliated, and satisfy the threshold property, then there is an optimal strategy that is regret-free.*

Proof. TBA. □

Intuitively, for positively affiliated projects, when a policy is frustration-free, the amount of attention that the DM is willing to allocate to one of the projects if there is no news is less than the amount of attention that he would allocate if the other project is completed. Thus, he is willing to allocate said attention independently of the outcome of the other project. The order in which the agent allocates this attention does not affect then the total payoff.

E Proofs of Section 6

E.1 Preliminaries

Let $\delta := \lambda^H - \lambda^L$. Using Bayes' rule, the beliefs $p_i(x_i)$, which denote the probability that the project i is easy, given that x_i cumulative attention was allocated to project i and project i was not completed, evolves according to:

$$p_i(x_i) = \frac{p_i e^{-\delta x_i}}{(1-p_i) + p_i e^{-\delta x_i}}$$

As the agent becomes more pessimistic, the subjective hazard rate $h_i(x_i)$ becomes lower.

$$h_i(x_i) = \lambda_L + p_i(x_i) \cdot \delta$$

For $x < \bar{x}$,

$$v_i(x) = \frac{1}{1-F(x)} \cdot \int_x^{\bar{x}} [1 - F(\tilde{x})] \cdot (h(\tilde{x}) \cdot q - c) \, d\tilde{x} \tag{6}$$

Next, we introduce two important lemmata: First, in Lemma 5, we prove that the projects are is sufficient to identify the monotonicity of project i's hazard-to-value ratio. Then, Lemma 6 shows that when hazard-to-value ratios are increasing or decreasing changes the sign of the determinant of the Hessian of the optimization problem $\max_X \hat{V}(X)$.

Lemma 5. *$h_i(x)/v_i(x)$ is monotone. Moreover, $h_i(x)/v_i(x)$ is increasing if and only if projects are cost effective when assorted.*

Proof. For this proof, we only focus on one of the projects, so we drop the subscript. Deriving Equation (6),

$$\begin{aligned} v'(x) &= \frac{f(x)}{[1-F(x)]^2} \int_x^{\bar{x}} [1 - F(\tilde{x})] \cdot (h(\tilde{x}) \cdot q - c) \, d\tilde{x} \; - \; (h(x) \cdot q - c) \\ &= h(x) \cdot v(x) + c - h(x) \cdot q \\ &= c - h(x) \cdot [q - v(x)]. \end{aligned}$$

Now we show that the monotonicity of $h(x)/v(x)$ depends on whether $v(x)$ is higher or lower than an expression $R(x)$.

$$\begin{aligned}
\operatorname{sgn}\left(\frac{\partial(h(x)/v(x))}{\partial x}\right) &= \operatorname{sgn}\left[h'(x)\cdot v(x) - h(x)\cdot v'(x)\right] \\
&= \operatorname{sgn}\left[h'(x)\cdot v(x) - h(x)\cdot [c - h(x)\cdot (q - v(x))]\right] \\
&= \operatorname{sgn}\left(\underbrace{\frac{h(x)\cdot [q\cdot h(x) - c\ ]}{h(x)^2 + h'(x)}}_{R(x)} \ - \ v(x)\right)
\end{aligned}$$

$$\frac{h(x)}{v(x)} \quad \text{decreasing} \qquad \Leftrightarrow \qquad R(x) < v(x) \tag{7}$$

It will be useful to switch to belief space.
Let $\hat{h} := \lambda_L + p\cdot\delta$. Notice that $\hat{h}(p(x)) = h(x)$. Then

$$h'(x) = \underbrace{\hat{h}'(p)}_{\delta}\cdot \underbrace{\frac{\partial p(x)}{\partial x}}_{-\delta\cdot p\cdot(1-p)}$$

$$\hat{R}(p) := \frac{\hat{h}(p)\cdot(q\cdot\hat{h}(p) - c)}{\hat{h}(p)^2 + \delta^2\cdot p\cdot(1-p)} \tag{8}$$

Deriving Equation (8) twice,

$$\hat{R}''(p) = \frac{2\ \delta^2\cdot\lambda_L\cdot\lambda_H\cdot(q\cdot\lambda_L\cdot\lambda_H - c\cdot(\lambda_L+\lambda_H))}{(\lambda_L^2 + p\ \delta\ (\lambda_L+\lambda_H))^3}$$

$$\begin{aligned}
\hat{R}''(p) > 0 \qquad &\Leftrightarrow \qquad q\cdot\lambda_L\cdot\lambda_H - c\cdot(\lambda_L+\lambda_H) > 0 \\
&\Leftrightarrow \qquad q > \frac{c}{\lambda_L} + \frac{c}{\lambda_H} \\
&\Leftrightarrow \qquad \text{project are not cost-effective when assorted.}
\end{aligned} \tag{9}$$

There are two cases to be considered separately: $\lambda_L < c$ and $\lambda_L \geqslant c$.

Case I: $\lambda_L\cdot q \geqslant c$ Since $h(x)\cdot q > \lambda_L\cdot q \geqslant c$, the agent that completes a project does never stop in the second stage. The value $\hat{v}$ is linear in the beliefs:

$$\hat{v}(p) = q - p \cdot \frac{c}{\lambda_H} - (1-p) \cdot \frac{c}{\lambda_L}$$

Since

$$\hat{v}(0) = q - \frac{c}{\lambda_L} = \hat{R}(0)$$

and

$$\hat{v}(1) = q - \frac{c}{\lambda_H} = \hat{R}(1)$$

If projects are effective when assorted, R is concave by Equation (9) and thus

$$v(p) < R(p) \qquad \forall p \in (0,1)$$

And then $h(x)/v(x)$ is increasing by Equation (8).

If, on the other hand, projects are not effective when assorted, R is convex by Equation (9) and thus

$$v(p) > R(p) \qquad \forall p \in (0,1)$$

And then $h(x)/v(x)$ is decreasing by Equation (8).

Case II: $\lambda_L \cdot q < c$ In this case, the agent stops putting attention to the remaining project if sufficient attention is allocated without success. More specifically, the agent will stop allocating attention to the remaining project when p reaches $\hat{p} := \frac{c/q-l}{h-l}$.

v is strictly convex (information is valuable). Moreover, we can show that R is concave:

$$\begin{aligned} \lambda_L \cdot q < c \qquad &\Rightarrow \qquad \frac{\lambda_H}{\lambda_L + \lambda_H} \cdot \lambda_L \cdot q < c \qquad \Leftrightarrow \qquad q < \frac{c}{\lambda_L} + \frac{c}{\lambda_H} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\, \Leftrightarrow \qquad \text{project are not cost-effective when assorted.} \end{aligned}$$

Since $\hat{v}(1) = q - \frac{c}{\lambda_H} = \hat{R}(1)$ and $v(\hat{p}) = 0 = R(\hat{p})$,

$$\hat{v}(p) < \hat{R}(p) \qquad \text{for any } p \in (\hat{p}, 1)$$

$$h(x)/v(x) \text{ decreasing}$$

$\square$

Lemma 6. *If $h_i(x)/v_i(x)$ is strictly decreasing for $i = A, B$, it is optimal to work on the projects in sequence.*

Proof. Since projects are complements and independent (thus positive affiliated), **??** indicates that there is an optimal first-stage strategy α that is regret-free. Let $X = X^\alpha$. By **??**, $\hat{V}_A(X) = \hat{V}_B(X) = V(X)$. Moreover, by optimality,

$$X \in \arg\max_{\tilde{X}} \hat{V}_i(\tilde{X}) \qquad \text{for } i = A, B$$

By contradiction assume that X is interior. Then the first-order conditions from deriving Equation (3) give us that

$$h_A(X_A) \cdot v_B(X_B) = h_B(X_B) \cdot v_A(X_A) = c \tag{10}$$

Claim: For an interior optimal point X,

$$\sum_{i=A,B} \frac{\partial h_i(X)/v_i(X)}{\partial X_i} > 0 \qquad \Rightarrow \qquad \prod_{i=A,B} \frac{h_i'(X_i) \cdot v_i(X_i)}{h_i(X_i) \cdot v_i'(X_i)} < 1$$

Using Equation (10),

$$\frac{h_A(X_A) \cdot v_A'(X_A)}{v_A^2(X_A)} = \frac{h_B(X_B) \cdot v_A'(X_A)}{v_B(X_B) \cdot v_A(X_A)} = \frac{h_A(X_A) \cdot v_B'(X_B)}{v_A(X_A) \cdot v_B(X_B)} = \frac{h_B(X_B) \cdot v_B'(X_B)}{v_B^2(X_B)}$$

Where the first and last equality use $h_A(X_A)/v_A(X_A) = h_B(X_B)/v_B(X_B)$ and the intermediate one uses that

$$\begin{aligned} h_B(X_B) \cdot v_A'(X_A) &= h_B(X_B) \cdot [c - h_A(X_A) \cdot (q - v_A(X_A)] \\ &= -h_B(X_B) \cdot h_A(X_A) \cdot [q - v_A(X_A) - v_B(X_B)] \end{aligned}$$

Since $c = h_A(X_A) \cdot v_B(X_B))$ and equal to $h_A(X_A) \cdot v_B'(X_B)$ by symmetry. So,

$$\begin{aligned}
\sum_{i=A,B} \frac{\partial h_i(X)/v_i(X)}{\partial X_i} > 0 \quad & \Leftrightarrow \quad \sum_{i=A,B} \frac{h_i'(X_i)\cdot v_i(X_i) - h_i(X_i)\cdot v_i'(X_i)}{v_i^2(X_i)} > 0 \\
& \Leftrightarrow \quad \sum_{i=A,B} \frac{h_i(X_i)\cdot v_i'(X_i)}{v_i^2(X_i)} \left(\frac{h_i'(X_i)\cdot v_i(X_i)}{h_i(X_i)\cdot v_i'(X_i)} - 1 \right) > 0 \\
& \Leftrightarrow \quad \left[\frac{h_A'(X_A)\cdot v_A(X_A)}{h_A(X_A)\cdot v_A'(X_A)} + \frac{h_B'(X_B)\cdot v_B(X_B)}{h_B(X_B)\cdot v_B'(X_B)} \right] < 2 \\
& \Leftrightarrow \quad \frac{h_A'(X_A)\cdot v_A(X_A)}{h_A(X_A)\cdot v_A'(X_A)} \cdot \frac{h_B'(X_B)\cdot v_B(X_B)}{h_B(X_B)\cdot v_B'(X_B)} < 1
\end{aligned}$$

Where the third implication uses that v_A is decreasing and the last one uses that the sum of two positive numbers being less than two implies that the product is less than one.

The determinant of the Hessian H for $V_A(X)$ is

$$\begin{aligned}
\det(H) \quad = \quad & [1-F_A(X_A)]\cdot[1-F_B(X_B)]\cdot \\
& \cdot [h_A'(X_A)\cdot h_B'(X_B)\cdot v_A(X_A)\cdot v_B(X_B) \; - \; h_A(X_A)\cdot h_B(X_B)\cdot v_A'(X_A)\cdot v_B'(X_B)]
\end{aligned}$$

So

$$\det(H) < 0 \quad \Leftrightarrow \quad \frac{h_A'(X_A)\cdot v_A(X_A)}{h_A(X_A)\cdot v_A'(X_A)} \cdot \frac{h_B'(X_B)\cdot v_B(X_B)}{h_B(X_B)\cdot v_B'(X_B)} < 1$$

And $\det(H) < 0$ rules implies that X is a saddle point, and thus not optimal. Thus it must be that the solution is not interior. $\square$

Proposition 4

- If the projects are not cost-effective when assorted, then it is efficient to work on them in sequence starting with the least promising one.
- If the projects are cost-effective when assorted, then it is efficient to work more on the most promising project.

Proof. We prove the two parts of the proposition separately.

Part I : If projects are not cost-effective when assorted, then by Lemma 5 we know that $h_i(x)/v_i(x)$ is decreasing for $i = A, B$. By Lemma 6 we know that it must be optimal to work on the projects in sequence. It remains to show that it is efficient to start with the least promising project.

Assume WLOG that $p_A > p_B$ and consider the alternative problem in which the initial beliefs are symmetric p_A for both projects. By symmetry, there must be two solutions $(X^*, 0)$ and $(0, X^*)$. Consider the second solution, the one that works on project B. The original problem is the same as the continuation problem of that problem after x attention was allocated to project B without success, where x is such that $p_A(x) = p_B$. Since the continuation strategy cannot be suboptimal, it must be that $(0, X^* - x)$ is a solution to the original problem. Thus, the agent works on the projects in sequence starting from the least promising one.

Part II :

$$\begin{aligned}
\frac{h_i(X_i)}{v_i(X_i)} \searrow \qquad & \Leftrightarrow \qquad h_i'(X_i) \cdot v_i(X_i) - h_i(X_i) \cdot v_i'(X_i) < 0 \\
& \Leftrightarrow \qquad \frac{h_i'(X_i) \cdot v_i(X_i)}{h_i(X_i) \cdot v_i'(X_i)} > 1
\end{aligned}$$

So,

$$\frac{h_i(X_i)}{v_i(X_i)} \searrow \text{ for } \quad i = A, B \qquad \Rightarrow \qquad \det(H) = \frac{h_A'(X_A) \cdot v_A(X_A)}{h_A(X_A) \cdot v_A'(X_A)} \cdot \frac{h_B'(X_B) \cdot v_B(X_B)}{h_B(X_B) \cdot v_B'(X_B)} > 1$$

This implies that there is at most one interior candidate for solution that satisfies the first order conditions $h_A(X_A) \cdot v_B(X_B) = h_B(X_B) \cdot v_A(X_A) = c$, and that if this candidate exist, it is the actual solution.

As before, assume WLOG that $p_A > p_B$ and consider the alternative problem in which the initial beliefs are symmetric p_A for both projects. By symmetry, the solution candidate is symmetrical (X^*, X^*). If $p_B > p_A(X^*)$, the original problem is the same as the continuation problem of that problem after x attention was allocated to project B without success, where x is such that $p_A(x) = p_B$. Since the continuation strategy cannot be suboptimal, it must be that $(X^*, X^* - x)$ is a solution to the original problem. If $p_B < p_A(X^*)$, the solution will not be interior. In that case, the agent works only on project i in the first stage with $h_i(X_i^*) \cdot v_j(0) = c > h_j(0) \cdot v_i(X_i^*)$. Since

$h_i(x)/v_i(x)$ is decreasing, it must be that $h_i(0)/v_i(0) > h_j(0)/v_j(0)$, what proves that i is the project with highest prior (since p is decreasing in x, $\hat{h}(p)/\hat{v}(p)$ is increasing).

$\square$